Online Appendix

A Derivation of the incentive compatibility condition (1)

First, we state formally the assumptions of the renegotiation game. If agent i renegotiates in state ω, it makes a take-or-leave-it offer to repay X instead of $\int d_j(\omega)\, dN^-_{ij} + a^-_i(\omega)$. A third party (e.g., a bankruptcy court) whose objective is to maximize the payment to agent i's claimholders, decides whether to accept the offer or not. If the offer is accepted, claimholders obtain X and the agent gets the value of its long positions in trees and Arrow securities minus X. We need to specify the payoffs of the agent and the claimholders when the offer is rejected and the agent's assets are seized. Because assets are imperfectly seizable, claimholders obtain fraction $1-\delta$ of the value of the agent's long positions. Moreover, we assume the agent gets a fraction $\kappa \in [0,1]$ of the payoffs of the fraction δ of the agent's long positions which creditors cannot seized. Hence, κ determines whether the part of collateral value that creditors cannot seize is a deadweight loss (if $\kappa = 0$) or diverted by the debtor ($\kappa = 1$) or any case in between. As will be clear below, the incentive compatibility condition does not depend on the value of κ.

Now, we determine the incentive compatibility condition (1) by requiring that liabilities are renegotiation proof. If the agent does not renegotiate, its payoff is assets minus repayment:

$$\left[\int d_j(\omega)\, dN^+_{ij} + a^+_i(\omega)\right] - \left[\int d_j(\omega)\, dN^-_{ij} + a^-_i(\omega)\right]. \tag{27}$$

If the agent renegotiates and offers to repay X, the offer is accepted if and only if the offered repayment is no less than the fraction $1-\delta$ of collateral value:

$$X \geq (1-\delta)\left[\int d_j(\omega)\, dN^+_{ij} + a^+_i(\omega)\right]. \tag{28}$$

Clearly, conditional on making an offer that is accepted, it is optimal for the agent to offer the smallest possible X that satisfies (28), $X = (1-\delta)\left[\int d_j(\omega)\, dN^+_{ij} + a^+_i(\omega)\right]$, leaving the agent with payoff:

$$\delta\left[\int d_j(\omega)\, dN^+_{ij} + a^+_i(\omega)\right]. \tag{29}$$

If the offer is rejected, then the payoff is $\kappa\,\delta \int d_j(\omega)\left[dN^+_{ij} + a^+_i(\omega)\right]$. Therefore, conditional on renegotiating, it is optimal for the agent to make an offer that is accepted. Combining (27) and (29), the agent does not renegotiate if and only if (1) holds.

B Proofs of the results in Section 3

B.1 Proof of Lemma 1

The proof follows the two steps outlined in the text. First, we show that the candidate equilibrium allocation $(\hat{c}_i, \hat{N}_i, \hat{a}_i)_{i\in I}$ clears markets. Second, we show that the plan $(\hat{c}_i, \hat{N}_i, \hat{a}_i)$ satisfies the incentive constraints of each agent. Third, we show that the plan $(\hat{c}_i, \hat{N}_i, \hat{a}_i)$ is budget feasible for all agents given prices (p, q). Taken together, these results show that the allocation $(\hat{c}_i, \hat{N}_i, \hat{a}_i)_i \in I$ is the basis of a financial-market equilibrium given prices (p, q).

Step 1: market clearing. As indicated in the text, we let $\hat{c}_i = c_i$. We also set $\hat{N}_i^- = 0$ and we scale down long tree positions so that tree markets clear. Formally, the existence of an appropriate scaling factor, $1-\theta_j$, follows from an application of the Radon-Nikodym Theorem (see, e.g., Royden and Fitzpatrick, 2010, page 382). Indeed, the market clearing condition, $\sum_{i\in I} N_i^+ = \bar{N} + \sum_{i\in I} N_i^-$, implies that $\bar{N}$ is absolutely continuous with respect to $\sum_{i\in I} N_i^+$. It thus follows from the Radon-Nikodym Theorem that there exists a measurable function $j \mapsto \theta_j \leq 1$ such that

$$d\bar{N}_j = \sum_{i\in I}(1-\theta_j)\, dN_{ij}^+. \tag{30}$$

Hence, if long tree positions are scaled down by $1-\theta_j$, i.e. if we let $d\hat{N}_{ij}^+ \equiv (1-\theta_j)\, dN_{ij}^+$, then the portfolios of long and short positions $(\hat{N}_i^+, \hat{N}_i^-)_{i\in I}$ clear the market for trees. Finally, we consider the Arrow securities positions

$$\hat{a}_i^+(\omega) = \max\left\{a_i^+(\omega) - a_i^-(\omega) + \int d_j(\omega)\left[\theta_j\, dN_{ij}^+ - dN_{ij}^-\right], 0\right\} \tag{31}$$

$$\hat{a}_i^-(\omega) = \max\left\{a_i^-(\omega) - a_i^+(\omega) + \int d_j(\omega)\left[dN_{ij}^- - \theta_j\, dN_{ij}^+\right], 0\right\}. \tag{32}$$

By construction, these satisfy

$$\hat{a}_i^+(\omega) - \hat{a}_i^-(\omega) + \int d_j(\omega)\left[d\hat{N}_{ij}^+ - d\hat{N}_{ij}^-\right] = a_i^+(\omega) - a_i^-(\omega) + \int d_j(\omega)\left[dN_{ij}^+ - dN_{ij}^-\right], \tag{33}$$

for all $\omega \in \Omega$, that is, the net security position (Arrow securities and trees) remain the same in all states. Using the market clearing for trees in the original allocation $(c_i, a_i, N_i)_{i\in I}$, (30) we obtain that

$$\sum_{i\in I}\left[dN_{ij}^+ - dN_{ij}^-\right] = \sum_{i\in I}(1-\theta_j)\, dN_{ij}^+ \Longrightarrow \sum_{i\in I}\left[\theta_j dN_{ij}^+ - dN_{ij}^-\right] = 0.$$

Combining this with the definition of $\hat{a}_i$, and using the market clearing for Arrow securities in the original equilibrium allocation, $(c_i, a_i, N_i)_{i\in I}$, we obtain market clearing for Arrow securities for the modified allocation, $(\hat{c}_i, \hat{a}_i, \hat{N}_i)_{i\in I}$.

Step 2: incentive compatibility. Next, we verify that all incentive constraints hold. To that end, notice that (1) can be rewritten:

$$\delta\left[\hat{a}_i^+(\omega) + \int d_j(\omega)\, d\hat{N}_{ij}^+\right] \leq \hat{a}_i^+(\omega) - \hat{a}_i^-(\omega) + \int d_j(\omega)\left[d\hat{N}_{ij}^+ - d\hat{N}_{ij}^-\right]. \tag{34}$$

Plugging in the definition of $\hat{a}_i^+(\omega)$ and $\hat{a}_i^-(\omega)$, in equations (31)-(32):

$$\begin{aligned}
\delta\left[\hat{a}_i^+(\omega) + \int d_j(\omega)\, d\hat{N}_{ij}^+\right] &= \delta\left[\max\left\{a_i^+(\omega) - a_i^-(\omega) + \int d_j(\omega)\left[\theta_j\, dN_{ij}^+ - dN_{ij}^-\right], 0\right\} + \int d_j(\omega)\,(1-\theta_j) dN_{ij}^+\right] \\
&\leq \delta\left[a_i^+(\omega) + \int d_j(\omega)\, dN_{ij}^+\right] \\
&\leq a_i^+(\omega) - a_i^-(\omega) + \int d_j(\omega)\left[dN_{ij}^+ - dN_{ij}^-\right] = \hat{a}_i^+(\omega) - \hat{a}_i^-(\omega) + \int d_j(\omega)\left[d\hat{N}_{ij}^+ - d\hat{N}_{ij}^-\right],
\end{aligned}$$

where: the equality on the first line follows by definition of the candidate equilibrium allocation, $(\hat{c}_i, \hat{a}_i, \hat{N}_i)$; the inequality on the second line follows because, for any $(a,b) \in \mathbb{R}^2$, $\max\{a+b, 0\} \leq \max\{a, 0\} + \max\{b, 0\}$; the inequality on the third line follows because the original equilibrium allocation satisfy all incentive constraint and because incentive constraints can be written as in (34); and the equality on the third line follows because the candidate equilibrium allocation keeps all net security positions the same as in the original equilibrium allocation, as shown in (33).

Step 3: budget feasibility. We first establish that agents do not find optimal to long (short) trees that are priced strictly above (below) replicating portfolio of Arrow securities. Formally, we let

$$J_0^+ \equiv \{j \in [0,1] \,:\, p_j > \sum_{\omega\in\Omega} q(\omega) d_j(\omega)\}$$
$$J_0^- \equiv \{j \in [0,1] \,:\, p_j < \sum_{\omega\in\Omega} q(\omega) d_j(\omega)\},$$

and we show that $N_i^+(J_0^+) = N_i^-(J_0^-) = 0$. We only state the proof that $N_i^+(J_0^+) = 0$, as the proof that $N_i^-(J_0^-) = 0$ is symmetric. Let us assume, towards a contradiction, that $N_i^+(J_0^+) > 0$. Then, consider an alternative plan that keeps the short positions the same, scales down the portfolio of over-priced trees, $j \in J_0^+$, by a factor $1-\lambda < 1$, and replace these trees by corresponding positions in their replicating portfolios of Arrow securities. Since these trees are over-priced, this generates a profit at time zero which can be used to increase consumption in each state. Formally, the alternative plan is $(\tilde{c}_i, \tilde{a}_i, \tilde{N}_i)$ such that $\tilde{a}_i^- = a_i^-$, $\tilde{N}_i^- = N_i^-$, $\tilde{a}_i^+(\omega) = a_i^+(\omega) + \lambda \int_{j\in J_0^+} d_j(\omega)\, dN_{ij}^+ + \Delta$, $d\tilde{N}_{ij}^+ = dN_{ij}^+ - \lambda \mathbb{I}_{\{j \in J_0^+\}} dN_{ij}^+$, where $\Delta > 0$ is chosen so that $\sum_{\omega\in\Omega} q(\omega)\Delta = \lambda \int_{j\in J_0^+} \left[p_j - \sum_{\omega\in\Omega} q(\omega) d_j(\omega)\right] dN_{ij}^+$. One easily sees that the incentive constraint (1) is relaxed, since, on the right-hand side, the total long security position is increased by Δ in all states while, on the left-hand side, the total short security position stays the same. Clearly, the alternative plan increases the agent's utility strictly, which contradicts optimality.

We now show that the candidate equilibrium allocation $(\hat{c}_i, \hat{N}_i, \hat{a}_i)$ constructed above is budget feasible at $t = 0$ for all $i \in I$. The left-hand side of the time-zero budget constraint evaluated at $(\hat{N}_i, \hat{a}_i)$ is:

$$\begin{aligned}
&\sum_{\omega\in\Omega} q(\omega) \left[\hat{a}_i^+(\omega) - \hat{a}_i^-(\omega)\right] + \int p_j \left[d\hat{N}_{ij}^+ - d\hat{N}_{ij}^-\right] \\
= \; &\sum_{\omega\in\Omega} q(\omega) \left[a_i^+(\omega) - a_i^-(\omega)\right] + \int p_j \left[dN_{ij}^+ - dN_{ij}^-\right] + \int \left[\sum_{\omega\in\Omega} q(\omega) d_j(\omega) - p_j\right] \left[\theta_j dN_{ij}^+ - dN_{ij}^-\right], \qquad (35)
\end{aligned}$$

where we used the expressions for $\hat{N}_i$ and $\hat{a}_i$. Notice that the first two terms in equation (35) correspond to the left-hand side of the time-zero budget constraint for the original plan (c_i, a_i, N_i). Hence, the time-zero budget constraint holds for the candidate plan $(\hat{c}_i, \hat{a}_i, \hat{N}_i)$ if the third term in equation (35) is negative. Using the definition of J_0^+ and J_0^-, as well as our preliminary result that $N_i^+(J_0^+) = 0$ and $N_i^-(J_0^-) = 0$, we obtain that this term can be written

$$-\int_{j\in J_0^+} \left[\sum_{\omega\in\Omega} q(\omega) d_j(\omega) - p_j\right] dN_{ij}^- + \int_{j\in J_0^-} \left[\sum_{\omega\in\Omega} q(\omega) d_j(\omega) - p_j\right] \theta_j dN_{ij}^+. \qquad (36)$$

Now we argue that both terms must be equal to zero. The intuition is simple. For the first term, it follows because agents take no long positions in trees $j \in J_0^+$, and so in an equilibrium they cannot take short position either. For the second term, it follows because agents take no short positions in trees $j \in J_0^-$, and so the scaling factor θ_j must be equal to zero.

Formally, consider the first term of equation (36). Given that $N_i^+(J_0^+) = 0$ and market clearing, we have $0 = \sum_{i\in I} N_i^+(J_0^+) = \bar{N}(J_0^+) + \sum_{i\in I} N_i^-(J_0^+)$. Hence, we obtain that $N_i^-(J_0^+) = 0$ for all i, so that the first term of (36) is equal to zero.

For the second term of (36), we first use market-clearing and the definition of θ_j to state that for all Borel sets A, $\sum_{i\in I} \int_{j\in A} (1-\theta_j) dN_{ij}^+ = \sum_{i\in I} \int_{j\in A} \left(dN_{ij}^+ - dN_{ij}^-\right)$. But, since $N_i^-(J_0^-) = 0$, it follows that $N_{ij}^-(A) = 0$ if $A \subseteq J_0^-$, which implies $\int_{j\in A} \theta_j \left(\sum_{i\in I} dN_{ij}^+\right) \geq 0$ for all Borel sets A, and with an equality if $A \subseteq J_0^-$. The inequality for all Borel sets implies that θ_j is positive almost everywhere according to $\sum_{i\in I} N_i^+$, and hence is positive almost everywhere according to N_i^+ for each $i \in I$. The equality for $A \subseteq J_0^-$ then implies that θ_j is zero almost everywhere over J_0^- according to N_i^+. It thus follows that the second term of (36) is equal to zero as well, and we are done.

B.2 Proof of Proposition 1

From financial market equilibrium to Arrow Debreu equilibrium. The first part of the Proposition, which states that any financial market equilibrium is the basis of an Arrow-Debreu equilibrium, is an implication of the following Lemma.

Lemma B.1 *A plan (c_i, N_i, a_i) such that $N_i^- = 0$ and $a_i^+(\omega)\, a_i^-(\omega) = 0$ satisfies the incentive compatibility constraint, (1), the time-zero budget constraint, (2), and the time-one budget constraint, (3), if and only if $a_i^+(\omega) = \max\left\{c_i(\omega) - \int d_j(\omega) dN_{ij}^+, 0\right\}$ and $a_i^-(\omega) = \max\left\{\int d_j(\omega) dN_{ij}^+ - c_i(\omega), 0\right\}$, and*

$$\sum_{\omega\in\Omega} q(\omega)c_i(\omega) + \int p_j\, dN_{ij} \leq \bar{n}_i \int p_j\, d\bar{N}_j + \sum_{\omega\in\Omega} q(\omega) \int d_j(\omega)\, dN_{ij}^+ \tag{37}$$

$$c_i(\omega) \geq \delta \int d_j(\omega)\, dN_{ij}^+ \text{ for all } \omega \in \Omega. \tag{38}$$

For the "only if" part, consider any plan (c_i, a_i, N_i) such that $N_i^- = 0$, $a_i^+(\omega)a_i^-(\omega) = 0$, that satisfies the incentive compatibility constraint (1), the time-zero budget constraint (2) and the time-one budget constraint (3). Taken together, the time-one budget constraint (3) and the assumption that $a_i^+(\omega)a_i^-(\omega) = 0$ imply the stated formulas for $a_i^+(\omega)$ and $a_i^-(\omega)$. The time-one budget constraint also implies that

$$a_i^+(\omega) - a_i^-(\omega) = c_i(\omega) - \int d_j(\omega)\, \left[dN_{ij}^+ - dN_{ij}^-\right]. \tag{39}$$

Subsituting this expression in the time-zero budget constraint, (3), we obtain the stated inter-temporal budget constraint (37). Lastly, given $N_i^- = 0$, the incentive constraint (1) implies the relaxed constraint $a_i^-(\omega) \leq a_i^+(\omega) + (1-\delta)\int d_j(\omega)\, dN_{ij}^+$. Substituting in this equation the expression for $a_i^+(\omega) - a_i^-(\omega)$ from (39), we obtain (38).

For the "if" part, consider a plan (c_i, N_i, a_i) satisfying (37), (38), $N_i^- = 0$, $a_i^+(\omega) = \max\left\{c_i(\omega) - \int d_j(\omega)dN_{ij}^+, 0\right\}$ and $a_i^-(\omega) = \max\left\{\int d_j(\omega)dN_{ij}^+ - c_i(\omega), 0\right\}$. By construction, the plan satisfies $N_i^- = 0$ and $a_i^+(\omega)a_i^-(\omega) = 0$ and is such that (39) holds. Hence, the time-one budget constraint, (3), holds. Substituting the expression for $c_i(\omega)$ given by the time-one budget constraint, (3), into the intertemporal budget constraint, (37), one then obtain the time-zero budget constraint (2). The only thing left to verify, then, is the incentive constraint (1). If $a_i^-(\omega) = 0$, then the agent has no liability and so the incentive constraint (1) holds trivially. If $a_i^-(\omega) > 0$, then $a_i^+(\omega) = 0$ and $a_i^-(\omega) = \int d_j(\omega)\, dN_{ij}^+ - c_i(\omega)$. Using this expression, we find that the incentive compatibility constraint (1) holds if and only if $\int d_j(\omega)\, dN_{ij}^+ - c_i(\omega) \leq (1-\delta)\int d_j(\omega)\, dN_{ij}^+$, which is the same as (38). The result follows.

From Arrow-Debreu equilibrium to financial market equilibrium. Let us consider an Arrow-Debreu equilibrium $(c_i, N_i^+)_{i\in I}$ and (p, q). Our candidate for a sequential market equilibrium is as follows. The plan for agent $i \in I$ is $\hat{c}_i(\omega) = c_i(\omega)$, $\hat{N}_i^+ = N_i$, $\hat{N}_i^- = 0$,

$$\begin{aligned}
\hat{a}_i^+(\omega) &= \max\left\{c_i(\omega) - \int d_j(\omega)\, dN_{ij}^+, 0\right\} \\
\hat{a}_i^-(\omega) &= \max\left\{\int d_j(\omega)\, dN_{ij}^+ - c_i(\omega), 0\right\}.
\end{aligned}$$

The price of trees is given by

$$\hat{p}_j = \min\left\{p_j, \sum_{\omega\in\Omega} q(\omega)d_j(\omega)\right\}$$

and the price of Arrow securities is given by $\hat{q}(\omega) = q(\omega)$. Notice that the price of trees coincides with p_j almost everywhere according to $\bar{N}$.

The candidate allocation clears all markets by construction. The tree market clearing condition implies that agents have zero positions in trees that are in zero supply. Hence, in the Arrow-Debreu equilibrium, the time-zero budget constraint holds for all agents when p is replaced by $\hat{p}$. Since, in addition, $\hat{N}_i^- = 0$ and $\hat{a}_i^+(\omega)\hat{a}_i^-(\omega) = 0$, it follows from an application of Lemma B.1 that $(\hat{c}_i, \hat{N}_i, \hat{a}_i)$ satisfies the time-zero and time one budget constraints, (2) and (3) given $(\hat{p}, \hat{q})$. The only thing that remains to be shown is that $(\hat{c}_i, \hat{N}_i, \hat{a}_i)$ is optimal given $(\hat{p}, \hat{q})$ in the agent's problem. This is not obvious because, once short-sales are allowed, the budget set in the Arrow-Debreu equilibrium is a subset of the budget set of the financial market equilibrium. That is, we need to show that the plan , we need to show that the plan $(\hat{c}_i, \hat{N}_i, \hat{a}_i)$ dominates any other plans, including the one prescribing short sales. To that end we anticipate the result of Proposition C.4 and let λ_i and $\{\mu_i(\omega)\}_{\omega\in\Omega}$ denote the Lagrange multipliers associated with $\hat{c}_i$ and $\hat{N}_i^+$, in the Arrow-Debreu agent's problem, . By construction, we have, for any $(\breve{c}_i, \breve{N}_i, \breve{a}_i)$ satisfying the time zero budget constraint, (2), the time one budget constraint, (3), and the incentive compatibility constraint, (1):

$$\lambda_i \left(\bar{n}_i \int \hat{p}_j \, d\bar{N}_j - \int \hat{p}_j \left[d\breve{N}_{ij}^+ - d\breve{N}_{ij}^- \right] - \sum_{\omega\in\Omega} \hat{q}(\omega) \left[\breve{a}_i^+(\omega) - \breve{a}_i^-(\omega) \right] \right) \geq 0$$

$$\pi(\omega) u_i'\left(\hat{c}_i(\omega)\right) \left(\int d_j(\omega) \left[d\breve{N}_{ij}^+ - d\breve{N}_{ij}^- \right] + \breve{a}_i^+(\omega) - \breve{a}_i^-(\omega) - \breve{c}_i(\omega) \right) \geq 0$$

$$\mu_i(\omega) \left((1 - \delta_i(\omega)) \breve{a}_i^+(\omega) + \int (1 - \delta_{ij}) d_j(\omega) \, d\breve{N}_{ij}^+ - \breve{a}_i^-(\omega) - \int d_j(\omega) \, d\breve{N}_{ij}^- \right) \geq 0,$$

with equalities if $(\breve{c}_i, \breve{N}_i, \breve{a}_i) = (\hat{c}_i, \hat{N}_i, \hat{a}_i)$. The standard optimality verification argument then implies:

$$\begin{aligned}
& U_i(\hat{c}_i) - U_i(\breve{c}_i) \\
\geq \; & \sum_{\omega\in\Omega} \pi(\omega) u_i'\left[\hat{c}_i(\omega)\right] \left(\hat{c}_i(\omega) - \breve{c}_i(\omega)\right) \\
+ \; & \lambda_i \left(- \int \hat{p}_j \left[d\hat{N}_{ij}^+ - d\hat{N}_{ij}^- \right] - \sum_{\omega\in\Omega} \hat{q}(\omega) \left[\hat{a}_i^+(\omega) - \hat{a}_i^-(\omega) \right] + \int \hat{p}_j \left[d\breve{N}_{ij}^+ - d\breve{N}_{ij}^- \right] + \sum_{\omega\in\Omega} \hat{q}(\omega) \left[\breve{a}_i^+(\omega) - \breve{a}_i^-(\omega) \right] \right) \\
+ \; & \sum_{\omega\in\Omega} \pi(\omega) u_i'\left[\hat{c}_i(\omega)\right] \left(\int d_j(\omega) \left[d\hat{N}_{ij}^+ - d\hat{N}_{ij}^- \right] + \hat{a}_i^+(\omega) - \hat{a}_i^-(\omega) - \hat{c}_i(\omega) - \int d_j(\omega) \left[d\breve{N}_{ij}^+ - d\breve{N}_{ij}^- \right] - \breve{a}_i^+(\omega) + \breve{a}_i^-(\omega) + \breve{c}_i(\omega) \right) \\
+ \; & \sum_{\omega\in\Omega} \mu_i(\omega) \Big((1 - \delta_i(\omega)) \hat{a}_i^+(\omega) + \int (1 - \delta_{ij}) d_j(\omega) \, d\hat{N}_{ij}^+ - \hat{a}_i^-(\omega) - \int d_j(\omega) \, d\hat{N}_{ij}^- \\
& \qquad - (1 - \delta_i(\omega)) \breve{a}_i^+(\omega) - \int (1 - \delta_{ij}) d_j(\omega) \, d\breve{N}_{ij}^+ + \breve{a}_i^-(\omega) + \int d_j(\omega) \, d\breve{N}_{ij}^- \Big) \\
= \; & \lambda_i \int \left(\hat{p}_j - v_{ij}\right) \left[d\breve{N}_{ij}^+ - d\hat{N}_{ij}^+ \right] + \lambda_i \int \left(\sum_{\omega\in\Omega} q(\omega) - \hat{p}_j \right) \left[d\breve{N}_{ij}^- - d\hat{N}_{ij}^+ \right] + \sum_{\omega} \mu_i(\omega) \delta_i(\omega) \left(\breve{a}_i^+(\omega) - \hat{a}_i^+(\omega) \right),
\end{aligned}$$

where the equality follows from collecting terms, substituting the expression for v_{ij} given in Proposition C.4, and using the first-order condition $\pi(\omega) u_i'\left[c_i(\omega)\right] + \mu_i(\omega) = \lambda_i \hat{q}(\omega)$ in Proposition C.4. Using the first-order conditions with respect to M_{ij} in Proposition C.4, keeping in mind that $\hat{p}_j \leq \sum_{\omega\in\Omega} q(\omega) d_j(\omega)$ by construction, and that $\mu_i(\omega) = 0$ whenever $\hat{a}_i^+(\omega) > 0$, we obtain that the right-hand side above is positive, implying that $U_i(\hat{c}_i) \geq U_i(\breve{c}_i)$. This concludes the proof.

C Proof of the results in Section 4

For the remainder of this appendix we prove all of our results for the generalized model in which δ depends on the agent and (continuously) on the tree type. It is easy to extend the argument of Proposition 1 and show that, when δ is agent- and tree-dependent, then any Arrow-Debreu equilibrium is the basis of a financial market equilibrium, with identical consumption allocation and security prices.

C.1 Proof of Proposition 2

That is, for each, $i \in I$, the function $j \mapsto \delta_{ij}$ is continuous. Let (c, N^+) denote an equilibrium allocation with associated price system (p, q). Suppose it is Pareto dominated by some other incentive-feasible allocation $(\hat{c}, \hat{N}^+)$. Then, because utility is strictly increasing, $\hat{c}_i$ must lie strictly outside the budget set of all agents for which $U_i(\hat{c}_i) > U_i(c_i)$. Otherwise, these agents would have a strict incentive to switch to $\hat{c}_i$. Likewise, $\hat{c}_i$ must lie weakly outside the budget set set of all agents for which $U_i(\hat{c}_i) = U_i(c_i)$. Otherwise, these agents would have strict incentive to increase their consumption in some state, which would respect incentive compatibility. Taken together, we obtain:

$$\sum_{\omega \in \Omega} q(\omega)\hat{c}_i(\omega) + \int p_j \, d\hat{N}^+_{ij} \geq \bar{n}_i \int p_j \, d\bar{N}_j + \int \sum_{\omega \in \Omega} q(\omega) d_j(\omega) \, d\hat{N}^+_{ij},$$

with a strict inequality for all $i \in I$ such that $U_i(\hat{c}_i) > U_i(c_i)$. Adding up across all agents we obtain that:

$$\sum_{\omega \in \Omega} q(\omega) \left\{ \sum_{i \in I} \hat{c}_i(\omega) - \sum_{i \in I} \int d_j(\omega) \, d\hat{N}^+_{ij} \right\} + \int p_j \left\{ \sum_{i \in I} d\hat{N}^+_{ij} - d\bar{N}_j \right\} > 0,$$

which contradicts the feasibility of $(\hat{c}, \hat{N})$.

C.2 Proof of Proposition 3

Our proof of existence proceeds as follows. In Section C.2.1 we define the Planner's Problem, we study some of its elementary properties, and we derive necessary and sufficient optimality conditions for a solution. In Section C.2.2, we turn to the equilibrium and derive first-order necessary and sufficient conditions for a solution to the agent's problem. Comparing the first-order conditions for the Planner and for the agent, in Section C.2.3 we show an equivalence between the set of equilibrium allocations, and the set of solutions to the Planner's problem with zero wealth transfers. We then establish the existence of a solution to the Planner's problem with zero wealth transfer. Omitted proofs are in Supplementary Appendix E.

In what follows we identify any measure with its cumulative distribution function. That is, we identify $\mathcal{M}_+$ with the set of increasing and right-continuous functions over $[0, 1]$. We denote by $\mathcal{M}$ the vector space of functions which can be written as $F = F_1 - F_2$, where both F_1 and F_2 belong to $\mathcal{M}_+$. We endow $\mathcal{M}$ with the total variation norm. Given any sequence $N^k \in \mathcal{M}$, we said that N^k *converges strongly* towards N , and write $N^k \to N$, if $\lim_{k \to \infty} \|N^k - N\| = 0$. We say that N^k *converges weakly* towards N, and write $N^k \Rightarrow N$, if $\int f_j \, dN^k_j \to \int f_j \, dN_j$ for all continuous real-valued functions $j \mapsto f_j$ over $[0, 1]$. A set of allocations K is said to be *weakly closed* if for any weakly converging sequence $(c^k, N^k) \in K$, i.e. such that $c^k \to c$ and $N^k \Rightarrow N$, then the limit of the sequence belongs to K, i.e., $(c, N) \in K$. The set K is said to be *weakly compact* if for any sequence $(c^k, N^k) \in K$, there exist some subsequence (c^ℓ, N^ℓ) and some $(c, N) \in K$ such that $c^\ell \to c$ and $N^\ell \Rightarrow N$.

C.2.1 The Planner's Problem

Let $\mathcal{A}$ denote the simplex, i.e., the set of welfare weights $\alpha \equiv (\alpha_1, \alpha_2, \ldots, \alpha_I)$ such that $\alpha_i \geq 0$ and $\sum_{i \in I} \alpha_i = 1$. Given any $\alpha \in \mathcal{A}$, and given any allocation (c, N^+), social welfare is defined as

$$W(\alpha, c, N) \equiv \sum_{i \in I} \alpha_i \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i(\omega)\right].$$

In the above formula, when $u_i(0) = -\infty$, we let $\alpha_i u_i \left[c_i(\omega)\right] = 0$ if $\alpha_i = c_i(\omega) = 0$.

Given weight $\alpha \in \mathcal{A}$, the *Planner's Problem* is:

$$W^\star(\alpha) = \sup W(\alpha, c, N^+) \tag{40}$$

with respect to incentive-feasible allocations, i.e., with respect to all allocations (c, N^+) satisfying (38), (9) and (8). We let $\Gamma^\star(\alpha)$ denote the set of allocations solving (40). To show the existence of a solution, we rely on:

Lemma C.1 *The set of incentive feasible allocations is weakly compact.*

The proof relies on Helly's Selection Theorem (Theorem 12.9 in Stokey and Lucas (1989)) which allows to extract weakly convergence subsequences from bounded sequences in $\mathcal{M}_+$. The feasibility and incentive compatibility constraints hold in the limit by definition of weak convergence. We add to the argument in Stokey and Lucas (1989) by showing that the feasibility constraint for tree holdings is also satisfied in the limit. With this result in mind, we show in the supplementary appendix:

Proposition C.1 *The planner's value $W^\star(\alpha)$ is a continuous function of $\alpha \in \mathcal{A}$, and the maximum correspondence $\Gamma^\star(\alpha)$ is non-empty, weakly compact, convex, and has a weakly closed graph. Moreover, consider any sequence $\alpha^k \to \bar{\alpha}$ and an associated sequence of optimal allocations $(c^k, N^{k+}) \in \Gamma^\star(\alpha^k)$. Then, if $\bar{\alpha}_i = 0$, $\lim_{k \to \infty} \alpha_i^k u' \left[c_i^k(\omega)\right] c_i^k(\omega) = 0$ for all $\omega \in \Omega$.*

If $u_i(0) = 0$ for all $i \in I$, the result follow from the same argument as in the proof of the Theorem of the Maximum (see, for example, Theorem 3.6 in Stokey and Lucas (1989)). If $u_i(0) = -\infty$ for some i, then we need to adapt the argument because the social welfare function is not continuous at (α, c, N) such that $\alpha_i = c_i(\omega) = 0$. Likewise, the result concerning $\alpha_i^k u' \left[c_i^k(\omega)\right] c_i^k(\omega) = 0$ is obvious if $u_i(0) = 0$, but requires some additional work when $u_i(0) = -\infty$.

To compare equilibria with solutions of the Planner's Problem, we rely on first-order conditions. We first derive necessary conditions. To do so, we cannot apply the Lagrange multiplier theorems of Luenberger (1969), because they do not accommodate equality constraints. Even if we consider a "relaxed problem" where equality constraints are replaced by inequality constraints, the theorems do not apply because the relevant positive cone has an empty interior. We therefore exploit the structure of the problem to derive first-order conditions by hand. To do so we consider, for any N, the maximized objective with respect to c. We then use an Envelope Theorem of Milgrom and Segal (2002) to explicitly calculate the directional derivative of this maximized objective with respect to N. We obtain:

Proposition C.2 *Suppose (c, N^+) solves the Planner's problem given some $\alpha \in \mathcal{A}$. Then there exists multipliers $\hat{q} \in \mathbb{R}_+^{|\Omega|}$ and $\hat{\mu} \in \mathbb{R}_+^{|\Omega| \times |I|}$ such that (c, N^+) satisfies two sets of conditions.*

- *First-order conditions:*

$$\begin{aligned} \alpha_i \pi(\omega) u_i' \left[c_i(\omega)\right] + \hat{\mu}_i(\omega) &= \hat{q}(\omega), \quad \forall (i, \omega) \in I \times \Omega \\ \int \left[\hat{p}_j - \hat{v}_{ij}\right] dN_{ij}^+ &= 0, \end{aligned}$$

  where $\hat{v}_{ij} \equiv \sum_{\omega \in \Omega} \hat{q}(\omega) d_j(\omega) - \sum_{\omega \in \Omega} \hat{\mu}_i(\omega) \delta_{ij} d_j(\omega)$, and $\hat{p}_j \equiv \max_{i \in I} \hat{v}_{ij}$.

- *Complementary slackness conditions:*

$$\begin{aligned}\hat{q}(\omega)\left[\sum_{i\in I}\int d_j(\omega)\,dN_{ij}^{+}-\sum_{i\in I}c_i(\omega)\right] &= 0 \quad \forall\omega\in\Omega\\ \hat{\mu}_i(\omega)\left[c_i(\omega)-\int\delta_{ij}d_j(\omega)\,dN_{ij}^{+}\right] &= 0 \quad \forall(i,\omega)\in I\times\Omega.\end{aligned}$$

Although the above conditions are also sufficient, it is convenient to state more general sufficient conditions, where $\hat{p}$ is taken to be some abstract continuous linear functional. This allows to show that any equilibrium is a solution to the Planner's Problem, even if the pricing functional cannot be represented by a continuous function. Then, using the necessary conditions derived in Proposition C.2, one can show that the same equilibrium allocation can be supported by a pricing functional represented by a continuous function.

Proposition C.3 *An incentive-feasible allocation (c, N^+) solves the Planner's problem if there exist multipliers $\hat{q}\in\mathbb{R}_+^{|\Omega|}$, $\hat{\mu}\in\mathbb{R}_+^{|\Omega|\times|I|}$, and a continuous linear functional $\hat{p}$ satisfying the following two sets of conditions.*

- *First-order conditions:*

$$\begin{aligned}\alpha_i\pi(\omega)u_i'\left[c_i(\omega)\right]+\hat{\mu}_i(\omega) &= \hat{q}(\omega), \quad \forall(i,\omega)\in I\times\Omega\\ \hat{p}\cdot M^{+}-\int\hat{v}_{ij}\,dM_{ij}^{+} &\geq 0 \quad \forall M_i^{+}\in\mathcal{M}_{+} \text{ and } i\in I, \text{ with ``='' if } M^{+}=N_i^{+},\end{aligned}$$

where $\hat{v}_{ij}\equiv\sum_{\omega\in\Omega}q(\omega)d_j(\omega)-\sum_{\omega\in\Omega}\mu_i(\omega)\delta_{ij}d_j(\omega)$.

- *Complementary slackness conditions:*

$$\begin{aligned}\hat{q}(\omega)\left[\sum_{i\in I}\int d_j(\omega)\,dN_{ij}^{+}-\sum_{i\in I}c_i(\omega)\right] &= 0 \quad \forall\omega\in\Omega\\ \hat{\mu}_i(\omega)\left[c_i(\omega)-\int\delta_{ij}d_j(\omega)\,dN_{ij}^{+}\right] &= 0 \quad \forall(i,\omega)\in I\times\Omega.\end{aligned}$$

C.2.2 Optimality conditions for the Agent's Problem

Notice that the range of the constraint set in the agent's problem is finitely dimensional. In this case, the "interior point condition" for the positive cone associated with the constraint set is immediately satisfied and so one can apply the general Lagrange multiplier theorems shown in Section 8.3 and 8.4 of Luenberger (1969).

Proposition C.4 *A plan (c_i, N_i^+) solve the agent's problem if and only if it satisfies the intertemporal budget constraint, (37), the incentive compatibility constraint (38), and there exists multipliers $\lambda_i\in\mathbb{R}_+$, $\mu_i\in\mathbb{R}_+^{|\Omega|}$ satisfying the following two sets of conditions:*

- *First-order conditions:*

$$\begin{aligned}\pi(\omega)u_i'\left[c_i(\omega)\right]+\mu_i(\omega) &= \lambda_i q(\omega)\\ \int\left(p_j-v_{ij}\right)dM_{ij}^{+} &\geq 0 \quad \forall M_i^{+}\in\mathcal{M}_{+}, \text{ with ``='' if } M_i^{+}=N_i^{+},\end{aligned}$$

where $v_{ij}\equiv\sum_{\omega\in\Omega}q(\omega)d_j(\omega)-\sum_{\omega\in\Omega}\frac{\mu_i(\omega)}{\lambda_i}\delta_{ij}d_j(\omega)$.

- *Complementary slackness conditions:*

$$\lambda_i \left[\bar{n}_i \int p_j \, d\bar{N}_j + \sum_{\omega \in \Omega} q(\omega) \int d_j(\omega) \, dN_{ij}^+ - \int p_j \, dN_{ij}^+ - \sum_{\omega \in \Omega} q(\omega) c_i(\omega) \right] = 0$$

$$\mu_i(\omega) \left[c_i(\omega) - \int \delta_{ij} d_j(\omega) \, dN_{ij}^+ \right] = 0 \quad \forall \omega \in \Omega.$$

There is one difference between this Proposition and the Theorems shown in Section 8.3 and 8.4 of Luenberger (1969): we are asserting that there exists multipliers such that the first-order condition with respect to $c_i(\omega)$ holds with equality. This follows from the following observation: if $c_i(\omega) = 0$, then the incentive compatibility constraint is binding, in particular $\int \delta_{ij} d_j(\omega) \, dN_{ij}^+ = 0$. Therefore, if we raise $\mu_i(\omega)$ so that the first-order condition holds with equality, we leave the product $\mu_i(\omega) \int \delta_{ij} d_j(\omega) \, dN_{ij}^+ = 0$ unchanged, which implies that $\int (p_j - v_{ij}) \, dN_{ij} = 0$ continues to hold. Finally, since raising $\mu_i(\omega)$ decreases v_{ij}, $\int (p_j - v_{ij}) \, dM_{ij}^+$ remains positive. Taken together, this means that we can always pick multipliers so that the first-order condition with respect to $c_i(\omega)$ holds with equality.

Finally, the following result provide a simple relationship between the value of the agent's endowment, and the marginal value of his consumption plan. This formula will be useful shortly to formulate the equilibrium fixed-point equation.

Lemma C.2 *If (c_i, N_i^+) solves the agent's problem, then*

$$\sum_{\omega \in \Omega} \pi(\omega) u' \left[c_i(\omega) \right] c_i(\omega) = \lambda_i \bar{n}_i \int p_j \, d\bar{N}_j.$$

The proof of the Lemma goes as follows. A solution to the agent's problem, (c_i, N_i^+), maximizes the Lagrangian:

$$\begin{aligned} L(\hat{c}_i, \hat{N}_i^+) = U_i(\hat{c}_i) \quad &+ \quad \lambda_i \left[\bar{n}_i \int p_j \, d\bar{N}_j + \sum_{\omega \in \Omega} q(\omega) \int d_j(\omega) \, d\hat{N}_{ij} - \int p_j \, d\hat{N}_{ij}^+ - \sum_{\omega \in \Omega} q(\omega) c_i(\omega) \right] \\ &+ \quad \sum_{\omega \in \Omega} \mu_i(\omega) \left[\hat{c}_i(\omega) - \int \delta_{ij} d_j(\omega) \, d\hat{N}_{ij} \right], \end{aligned}$$

with respect to $(\hat{c}_i, \hat{N}_i)$. This implies that the function $\beta \mapsto L(\beta c_i, \beta N_i)$ is maximized at $\beta = 1$. Taking first-order condition with respect to β at $\beta = 1$, and using the complementary slackness conditions, yields the desired result.

C.2.3 Existence of a Planner's Solution with Zero Wealth Transfer

By comparing the first-order conditions of the Planner and of the agent, we obtain:

Proposition C.5 *An allocation (c, N^+) is an Arrow-Debreu equilibrium allocation if and only if there exists $\alpha \in \mathcal{A}$ such that:*

- *(c, N^+) solves the Planner's problem given α;*
- *For all $i \in I$, $\alpha_i \sum_{\omega \in \Omega} \pi(\omega) u_i' \left[c_i(\omega) \right] c_i(\omega) = \bar{n}_i \sum_{k \in I} \sum_{\omega \in \Omega} \pi(\omega) u_k' \left[c_k(\omega) \right] c_k(\omega)$.*

In particular, given a solution of the Planner's problem satisfying the above two conditions, an equilibrium price system is given by the multipliers $(\hat{p}, \hat{q})$ of Proposition C.2.

Intuitively, comparing the first-order conditions of the Planner and of the agent reveals that the weight α_i must be proportional to $1/\lambda_i$, the inverse of the Lagrange multiplier on the agent's budget constraint. It then follows from

Lemma C.2 that, for all agents $i \in I$:

$$\alpha_i \sum_{\omega\in\Omega} \pi(\omega) u_i' \left[c_i(\omega)\right] = \bar{n}_i \times \left[\sum_{k\in I} \frac{1}{\lambda_k}\right]^{-1} \times \int p_j \, d\bar{N}_j.$$

The second condition then follows because $\sum_{i\in I} \bar{n}_i = 1$. The final result about the price system follows from direct comparison of the first-order conditions of the agent and the planner.

We are now ready to establish the existence of an equilibrium. Let $\Delta^\star(\alpha)$ denote the set of transfers $\{\Delta_i^\star(\alpha)\}_{i\in I}$ such that:

$$\Delta_i^\star(\alpha) \equiv \alpha_i \sum_{\omega\in\Omega} \pi(\omega) u_i' \left[c_i(\omega)\right] c_i(\omega) - \bar{n}_i \sum_{k\in I} \alpha_k \sum_{\omega\in\Omega} \pi(\omega) u_k' \left[c_k(\omega)\right] c_k(\omega), \tag{41}$$

generated by all $(c, N^+) \in \Gamma^\star(\alpha)$, with the convention that $\alpha_i u_i'(c)c = 0$ if $\alpha_i = c = 0$. Using the Kakutani's fixed-point Theorem, as in Negishi (1960) and Magill (1981), we can show:

Proposition C.6 *There exists some $\alpha \in \mathcal{A}$, such that $0 \in \Delta^\star(\alpha)$.*

Based on some $\alpha \in \mathcal{A}$, using Proposition C.5, we can construct an equilibrium allocation and price system.

C.3 First Best Implementability

The objective of this appendix is to study circumstances under which the incentive compatibility constraints do not impact equilibrium outcomes. Formally, define a $\delta = 0$ equilibrium to be a first-best allocation and price system (c^0, N^{0+}, p^0, q^0) when $\delta = 0$, i.e., when there is no scope for incentive problems. Fix some $\delta > 0$. Then, the $\delta = 0$-equilibrium is said to be $\delta > 0$-*implementable* if there exists some $\delta > 0$-equilibrium, $(c^\delta, N^{\delta+}, q^\delta, p^\delta)$, such that $c^0 = c^\delta$. It is clear that a $\delta = 0$-equilibrium, (c^0, N^{0+}, p^0, q^0), is $\delta > 0$-implementable if and only if there exists a feasible tree allocation, $N^{\delta+}$, such that incentive constraints are satisfied for all agents and all states:

$$\sum_{i\in I} N_i^{\delta+} = \bar{N} \tag{42}$$

$$c_i^0(\omega) \geq \delta \int d_j(\omega) \, dN_{ij}^{\delta+} \quad \forall (i,\omega) \in I \times \Omega. \tag{43}$$

The feasibility condition (42) is crucial: in its absence, the incentive constraint would not have any bite, since it would be trivial to satisify (43) by setting $N_i^+ = 0$ for all agents. This observation means that, in our model, binding incentive compatibility constraints is ultimately a general equilibrium phenomenon. This is in contrast with earlier models in which non-pledgeable income is not tradeable: in these environments, binding incentive constraints would already arise in partial equilibrium contract-theoretic settings.

Using conditions (42) and (43), we obtain:

Proposition C.7 *Fix some $\delta > 0$. A $\delta = 0$ equilibrium (c^0, N^{0+}, p^0, q^0) is $\delta > 0$-implementable if one of the following conditions is satisfied:*

- *Inada conditions are satisfied for all $i \in I$ and δ is strictly positive but small enough.*
- *There exists $\{N_i^+\}_{i\in I} \in \mathcal{M}_+^{|I|}$ such that $\sum_{i\in I} N_i^+ = \bar{N}$ and $\int d_j(\omega) \, dN_{ij}^+ = c_i^0(\omega)$ $\forall (i,\omega) \in I \times \Omega$.*
- *Agents have Constant Relative Risk Aversion (CRRA) with identical coefficient.*

To understand the first bullet point, note that, with Inada conditions, consumptions are strictly positive for all agents and all states. Therefore, as long as δ is small enough, the incentive compatibility constraint (43) is satisfied for all agents and all states when each agent holds, say, an equal fraction of the market portfolio, $N_i^+ = \bar{N}/|I|$, and simultaneously issues liabilities to attain its first-best consumption plan.

The second bullet point of the proposition states that all incentive compatibility constraints hold if two sets of conditions are satisfied. First agents can replicate their $\delta = 0$-equilibrium consumption with *positive* holdings of trees. Second, these agents' holdings are *feasible*, i.e., they add up to the aggregate. This means that they do not need to make any financial promise, i.e., promise to deliver consumption out of the payoff of their equilibrium holdings of trees. Clearly, if agents do not need to make any financial promise, limited pledgeability is not an issue.

The third bullet point is an example of the second: if agents have CRRA utilities with identical risk aversion, then they all consume a constant share of the aggregate endowment. Clearly, they can attain that consumption plan by holding a portfolio of trees, namely a constant share in the market portfolio.

Equations (42)-(43) and Proposition C.7 also help understand circumstances under which a $\delta = 0$ equilibrium *cannot* be implemented. Consider for example an economy composed of CRRA utility agents with heterogenous risk aversion, and assume that there is only one tree, the "market portfolio", with payoff equal to aggregate consumption. Because of heterogeneity in risk aversion, in the $\delta = 0$ equilibrium, agents' consumption shares vary across states – namely more risk averse agents tend to have higher consumption shares in states of low aggregate consumption. If $\delta \simeq 1$, agents cannot issue liabilities and their consumption must be approximately equal to the payoff of their tree portfolio. But since they can only hold the market portfolio, their consumption share must be approximately constant across states, so that the $\delta \simeq 1$ equilibrium cannot coincide with the $\delta = 0$ equilibrium.

C.4 Uniqueness in the CRRA ≤ 1 case

Proposition C.8 *Suppose that there are two types of agents, $I = \{1, 2\}$, with CRRA utility, with respective RRA coefficients (γ_1, γ_2) such that $0 \leq \gamma_1 \leq \gamma_2 \leq 1$ and $\gamma_2 > 0$. Then the Arrow-Debreu equilibrium consumption allocation is uniquely determined. The prices of consumption claims, q, the price of trees, p, are uniquely determined $\bar{N}$-almost everywhere up to a positive multiplicative constant.*

In general, the tree allocation is not uniquely determined. As will be clear below, this arises for example when none of the incentive constraints bind. In that case the allocation is not uniquely determined because it is equivalent to hold tree j or a portfolio of Arrow securities with the same cash-flows as j.

As is standard, only relative prices are pinned down, hence price levels are only determined up to a positive multiplicative constant.

In an Arrow-Debreu equilibrium, tree prices are only uniquely determined $\bar{N}$-almost everywhere. In particular, the prices of trees in zero net supply are not uniquely determined. This is intuitive: given the short-sale constraint, the only equilibrium requirement for a tree in zero supply is that the price is large enough so that no agent want to hold it. As a result equilibrium only imposes a lower bound on the price of trees in zero supply.[24]

Step 1: an equivalent one-equation-in-one-unknown problem. Since the utility function of agent $i = 2$ is strictly concave, its allocation is uniquely determined in the Planner's problem. But since $c_1(\omega) + c_2(\omega) = \int d_j(\omega)\, d\bar{N}_j$, the consumption allocation of agent 1 is also uniquely determined. Hence $\Delta^\star(\alpha)$, defined in equation (41), is a function and not a correspondence. Moreover since $\Delta_1^\star(\alpha) + \Delta_2^\star(\alpha) = 0$ by construction and $\alpha_1 + \alpha_2 = 1$ by assumption, it is enough to look for a solution of $\Delta_1^\star(\alpha_1, 1 - \alpha_1) = 0$. That is, solving for equilibrium boils down to a one-equation in one-unknown problem. To formulate this problem in simple terms, let

$$\mathrm{MU}_i(c_i) \equiv \sum_\omega \pi(\omega) u_i' \left[c_i(\omega)\right] c_i(\omega).$$

[24] To remove the indeterminacy, it would be natural to inject a small additional supply for all trees, $\bar{N} + \varepsilon\,\mathcal{U}_{[0,1]}$ and let $\varepsilon \to 0$. In addition, as mentioned earlier, the implementation of an Arrow equilibrium as a security market equilibrium rules out very high prices, and so reduces the indeterminacy.

Notice, that with CRRA utility, $\text{MU}_i(c_i) = (1-\gamma_i)U_i(c_i)$ for $\gamma_i \neq 1$, and $\text{MU}_i(c_i) = 1$ for $\gamma_i = 1$. With this notation, the one-equation-in-one-unknown problem for equilibrium is:

$$\bar{n}_2\alpha_1\text{MU}_1(c_1) - \bar{n}_1\alpha_2\text{MU}_2(c_2) = 0, \tag{44}$$

where (c_1, c_2) is the consumption allocation chosen by the planner given weight $\alpha \in \mathcal{A}$. We already know from Proposition C.6 that this equation has a solution.

Step 2: an intermediate result. An intermediate result for proof of uniqueness is the observation that, for any α' and α such that $\alpha'_1 > \alpha_1$, and for all for all $c \in \Gamma^\star(\alpha)$ and $c' \in \Gamma^\star(\alpha')$.

$$U_1(c'_1) \geq U_1(c_1) \text{ and } U_2(c'_2) \leq U_2(c_2)$$
$$\text{MU}_1(c'_1) \geq \text{MU}_1(c_1) \text{ and } \text{MU}_2(c'_2) \leq \text{MU}_2(c_2)$$

To prove this intermediate result, consider two sets of weights α and α' with corresponding optimal allocations $(c, N^+) \in \Gamma^\star(\alpha)$ and $(c', N'^+) \in \Gamma^\star(\alpha')$. Since the constraint set of the planner does not depend on α, (c, N^+) and (c', N'^+) are both incentive feasible given α and α'. Hence, optimality implies that:

$$\alpha_1 U_1(c_1) + \alpha_2 U_2(c_2) \geq \alpha_1 U_1(c'_1) + \alpha_2 U_2(c'_2) \Leftrightarrow \alpha_1\left[U_1(c_1) - U_1(c'_1)\right] + \alpha_2\left[U_2(c_2) - U_2(c'_2)\right] \geq 0.$$

Vice versa:

$$\alpha'_1\left[U_1(c'_1) - U_1(c_1)\right] + \alpha'_2\left[U_2(c'_2) - U_2(c_2)\right] \geq 0.$$

Adding up these two inequality and using that, since the weight add up to one, $\alpha'_1 - \alpha_1 = \alpha_2 - \alpha'_2$, we obtain:

$$\left[\alpha'_1 - \alpha_1\right]\left\{\left[U_1(c'_1) - U_1(c_1)\right] - \left[U_2(c'_2) - U_2(c_2)\right]\right\},$$

which implies that:

$$U_1(c'_1) - U_1(c_1) \geq U_2(c'_2) - U_2(c_2).$$

But then we must have that

$$U_1(c'_1) - U_1(c_1) \geq 0 \geq U_2(c'_2) - U_2(c_2).$$

because otherwise either (c, N) or (c', N') would not be constrained Pareto optima.

Step 3: equation (44) has a unique solution. We now go back to equation (44). Let α denote some solution, and consider any $\alpha' \neq \alpha$, for example such that $\alpha'_1 > \alpha_1$. Let c and c' denote the consumption allocations associated with α and α'. Then,

$$\begin{aligned}
& \bar{n}_2\alpha'_1\text{MU}_1(c'_1) - \bar{n}_1\alpha'_2\text{MU}_2(c'_2) \\
= \quad & \bar{n}_2\alpha'_1\text{MU}_1(c'_1) - \bar{n}_1\alpha'_2\text{MU}_2(c'_2) - \bar{n}_2\alpha_1\text{MU}_1(c_1) + \bar{n}_1\alpha_2\text{MU}_2(c_2) \\
= \quad & \bar{n}_2\alpha'_1\left[\text{MU}_1(c'_1) - \text{MU}_1(c_1)\right] - \bar{n}_1\alpha'_2\left[\text{MU}_2(c'_2) - \text{MU}_2(c_2)\right] + \left(\alpha'_1 - \alpha_1\right)\left[\bar{n}_2\text{MU}_1(c_1) + \bar{n}_1\text{MU}_2(c_2)\right] > 0.
\end{aligned}$$

In the above, the second line follows from subtracting $\bar{n}_2\alpha_1\text{MU}_1(c_1) - \bar{n}_1\alpha_2\text{MU}_2(c_2) = 0$ since α was assumed to solve (44). The third line follows from re-arranging terms and keeping in mind that $\alpha'_1 - \alpha_1 = \alpha_2 - \alpha'_2$. The inequality follows from the intermediate result established in Step 2, and from the fact that marginal utilities are strictly positive. Vice versa, if we consider some $\alpha' \neq \alpha$ such that $\alpha'_1 < \alpha_1$, we obtain that the equilibrium equation

(44) is strictly negative. Therefore, the equation for the weight, α, has a unique solution.

Step 4: the various uniqueness claims. Consider any equilibrium allocation, (c, N^+), and price system, (p, q). From Proposition C.5, we know that (c, N^+) solves the Planner's given the unique set of weights such that $\Delta^\star(\alpha) = 0$. But, as argued above, the consumption allocation is uniquely determined in the Planner's problem. Hence, it follows that the equilibrium allocation is uniquely determined in an equilibrium as well. Next, by direct comparison of first-order conditions, one sees that (c, N) solve the first-order conditions of the Planner's problem with weights $\alpha_i = \beta/\lambda_i$, multipliers $\hat{q}(\omega) = \beta q(\omega)$, $\hat{\mu}_i(\omega) = \alpha_i \mu_i(\omega)$, $\hat{v}_{ij} = \beta v_{ij}$ and $\hat{p}_j = \beta p_j$, where λ_i is the Lagrange multiplier on agent's i budget constraint, and $\beta \equiv \left[\sum_{k \in I} 1/\lambda_k\right]^{-1}$. But from the first-order conditions of the Planner's problem, and given that c is uniquely determined, it follows that $\hat{q}(\omega)$, $\hat{\mu}(\omega)$ and $\hat{v}_{ij}$ are uniquely determined as well. Clearly, this implies that the price of Arrow securities, q, and the private tree valuations, v, are uniquely determined up to the multiplicative constant $1/\beta$. Now turning to the price of trees, we note that the first-order condition of the agent's problem imply that $p_j = v_{ij}$ for almost all trees held by i. Since the private valuations are uniquely determined up to the multiplicative constant $1/\beta$, the same property must hold for the price trees $\bar{N}$-almost everywhere.

C.5 Proof of Proposition 4

Suppose the equilibrium is not first best. Consider an agent i. There must exist some state in which agent i's incentive constraint binds, otherwise it would be possible to reallocate a small fraction of all tree supplies to agent i while still satisfying his incentive constraint, and that would make all other agents' incentive constraints slack, contradicting that the equilibrium is not first best.

Consider state ω in which agent i's incentive constraint binds. There must exist some other agent i' whose incentive constraint is slack in state ω. Indeed, adding up the incentive constraint (6) across all agents and using market clearing for consumption, one immediately sees that the aggregate incentive constraint must be slack in each state. Consider $\varepsilon > 0$ and tree j such that $d_j(\omega')/d_j(\omega) < \varepsilon$ for all $\omega' \neq \omega$. Then:

$$
\begin{aligned}
p_j &\geq \sum_{\omega' \in \Omega} q(\omega') d_j(\omega') - \sum_{\omega' \in \Omega} \frac{\mu_{i'}(\omega')}{\lambda_{i'}} \delta d_j(\omega') \\
&= \sum_{\omega' \in \Omega} q(\omega') d_j(\omega') - \sum_{\omega' \neq \omega} \frac{\mu_{i'}(\omega')}{\lambda_{i'}} \delta d_j(\omega') \\
&\geq \sum_{\omega' \in \Omega} q(\omega') d_j(\omega') - \sum_{\omega' \neq \omega} \frac{\mu_{i'}(\omega')}{\lambda_{i'}} \varepsilon \delta d_j(\omega) \\
&> \sum_{\omega' \in \Omega} q(\omega') d_j(\omega') - \frac{\mu_i(\omega)}{\lambda_i} \delta d_j(\omega) \\
&\geq \sum_{\omega' \in \Omega} q(\omega') d_j(\omega') - \sum_{\omega' \in \Omega} \frac{\mu_i(\omega')}{\lambda_i} \delta d_j(\omega') \\
&= v_{ij},
\end{aligned}
$$

where the first line is equation (12) for agent i, the second line obtains because agent i's incentive constraint is slack in state ω, the third line obtains because $d_j(\omega')/d_j(\omega) < \varepsilon$ for all $\omega' \neq \omega$, the fourth line obtains by picking some ε such that $\varepsilon < \frac{\mu_i(\omega)}{\lambda_i} / \sum_{\omega' \neq \omega} \frac{\mu_{i_0}(\omega')}{\lambda_i}$, the fifth line obtains because $\mu_i(\omega') \geq 0$ for all ω', and the sixth line is equation (12) for agent i.

C.6 Proof of Proposition 5

Consider agent i holding tree j. His valuation for the tree must equal the tree price:

$$\begin{aligned} p_j &= \sum_{\omega\in\Omega} q(\omega)d_j(\omega) - \sum_{\omega\in\Omega}\frac{\mu_i(\omega)}{\lambda_i}\delta d_j(\omega) \\ &= \sum_{\omega\in\Omega}\left[q(\omega)-\frac{\mu_i(\omega)}{\lambda_i}\delta\right]\left[\int d_k(\omega)dX_k + Y(\omega)\right] \\ &= \int v_{ik}dX_k + \sum_{\omega\in\Omega}\left[q(\omega)-\frac{\mu_i(\omega)}{\lambda_i}\delta\right]Y(\omega) \\ &< \int p_k dX_k + \sum_{\omega\in\Omega} q(\omega)Y(\omega), \end{aligned}$$

where we move to the second line by substituting the tree payoff with the payoff of the replicating portfolio, we move to the third line using expression (12) for agent i's valuation for trees k, and we move to the fourth line using that agent i is not willing to hold all assets in the replicating portfolio, which implies $\int v_{ik}dX_k < \int p_k dX_k$ or $\mu_i(\omega)Y(\omega) > 0$ for some ω.

D Proof of the results in Section 5

D.1 Proof of Lemma 2

As before we state proofs for our results when δ_{ij} is assumed to depend both on the type of agent holding the tree and on the type of the tree. In this case, the Proposition holds under the additional restriction that:

$$\frac{\delta_{1j}d_j(\omega_1)}{\delta_{2j}d_j(\omega_2)}, \tag{45}$$

is strictly increasing. Notice that this restriction is automatically satisfied whenever $\delta_{1j} = \delta_{2j}$ for all j. The generalization of (20)-(21) is

$$c_1(\omega_1) \geq \int_{j\in[0,k)} \delta_{1j}d_j(\omega_1)d\bar{N}_j + \delta_{1k}d_k(\omega_1)\Delta N_1 \tag{46}$$

$$c_2(\omega_2) \geq \int_{j\in(k,1]} \delta_{2j}d_j(\omega_2)d\bar{N}_j + \delta_{2k}d_k(\omega_2)\Delta N_2. \tag{47}$$

The "if" part of the Proposition. Pick the smallest possible k and the largest possible ΔN_2 such that the inequalities (46)-(47) hold. Consider the corresponding tree allocation $N_1 = \bar{N}\mathbb{I}_{\{j\in[0,k)\}} + \Delta N_1\mathbb{I}_{\{j=k\}}$ and $N_2 = \Delta N_2\mathbb{I}_{\{j=k\}} + \bar{N}\mathbb{I}_{\{j\in(k,1]\}}$.

By construction, the incentive constraint of agent $i = 1$ holds in state ω_1, and the incentive constraint of agent $i = 2$ holds in state ω_2.

Next, we argue that the incentive constraint of agent $i = 1$ holds in state ω_2. This is obvious if N allocates no tree to agent $i = 1$. Otherwise, if N allocates some trees to agent $i = 1$, then the incentive constraint of agent $i = 2$ must bind in state ω_2. Given $\delta_{ij} < 1$, this implies that the incentive constraint of agent $i = 1$ holds in state ω_2.

With the above observations in mind, the only incentive constraint that remains to be checked is that of agent

$i = 2$ in state ω_1. If it holds with the proposed tree allocation, N, we are done. Otherwise,

$$c_2(\omega_1) < \int_{(k,1]} \delta_{2j} d_j(\omega_1)\, d\bar{N}_j + \delta_{2k} d_{2k} \Delta N_2,$$

in which case we explicitly construct another allocation of tree holdings that is incentive compatible. We proceed as follows. We start from the proportional tree allocation that delivers agents $i = 1$ and $i = 2$ their consumption in state ω_2: $\hat{N}_1 = \frac{c_1(\omega_2)}{y(\omega_2)} \bar{N}$ and $\hat{N}_2 = \frac{c_2(\omega_2)}{y(\omega_2)} \bar{N}$. By construction, with such proportional allocation, the incentive constraint of both agents hold in state ω_2. Since the consumption share of agent $i = 2$ is strictly larger in state ω_1 than in state ω_2, one sees that that agent $i = 2$ incentive compatibility constraint is slack in state ω_1. Indeed, we have:

$$c_2(\omega_1) > \frac{y(\omega_1)}{y(\omega_2)} c_2(\omega_2) = \frac{c_2(\omega_2)}{y(\omega_2)} \int d_j(\omega_1) d\bar{N}_j = \int d_j(\omega_1) d\tilde{N}_{2j} > \int \delta_{2j} d_j(\omega_1) d\tilde{N}_{2j},$$

where the first inequality states that the consumption share is larger in state ω_1 than in state ω_2, the first equality follows from rearranging and from the definition of $y(\omega_1)$, the second equality follows from the definition of N_2, and the last inequality follows because $\delta_{2j} < 1$.

Taking stock, for the original allocation N, the incentive compatibility constraints hold in state ω_2 for both $i = 1$ and $i = 2$, but it does not hold in state ω_1 for agent $i = 2$. For the proportional allocation $\hat{N}$, the incentive compatibility constraints also hold in state ω_2 for both $i = 1$ and $i = 2$, and it holds with strict inequality in state ω_1 for agent $i = 2$. Therefore, there is a convex combination of N and $\hat{N}$ such that the incentive compatibility constraint is binding in state ω_1 for agent $i = 2$. This implies that the incentive compatibility constraint holds in state ω_1 for agent $i = 1$. Clearly, the incentive compatibility constraint also hold in state ω_2 for both agents since they hold separately for N and $\hat{N}$.

The "only if" part of the Proposition. As before, pick the smallest possible k and the largest possible ΔN_2 such that (47) holds. Given this ΔN_2, let $\Delta N_1 \equiv \hat{N}_k - \bar{N}_{k-} - \Delta N_2$. If $k = 0$ and $\Delta N_2 = \bar{N}_0$, then (46) evidently holds. Otherwise, (47) holds with equality and we need to establish that that (46) holds as well. To that end, consider any $\hat{N}$ such that $(c, \hat{N})$ is incentive feasible. Then:

$$\begin{aligned}
& \int_{[0,k)} \delta_{1j} d_j(\omega_1) d\bar{N}_j + \delta_{1k} d_k(\omega_1) \Delta N_1 \\
= \;& \int_{[0,k)} \delta_{1j} d_j(\omega_1) \left(d\hat{N}_{1j} + d\hat{N}_{2k} \right) + \delta_{1k} d_k(\omega_1) \Delta N_1 \\
= \;& \int_{[0,1]} \delta_{1j} d_j(\omega_1) d\hat{N}_{1j} - \int_{[k,1]} \delta_{1j} d_j(\omega_1) d\hat{N}_{1j} + \delta_{1k} d_k(\omega_1) \Delta N_1 + \int_{[0,k)} \delta_{1j} d_j(\omega_1) d\hat{N}_{2j} \\
\leq \;& c_1(\omega_1) - \int_{[k,1]} \delta_{2j} d_j(\omega_2) \frac{\delta_{1k} d_k(\omega_1)}{\delta_{2k} d_k(\omega_2)} d\hat{N}_{1j} + \delta_{1k} d_k(\omega_1) \Delta N_1 + \int_{[0,k)} \delta_{2j} d_j(\omega_2) \frac{\delta_{1k} d_k(\omega_1)}{\delta_{2k} d_k(\omega_2)} d\hat{N}_{2j} \\
= \;& c_1(\omega_1) + \frac{\delta_{1k} d_k(\omega_1)}{\delta_{2k} d_k(\omega_2)} \left[\int_{[0,1]} \delta_{2j} d_j(\omega_2) d\hat{N}_{2j} + \delta_{2k} d_k(\omega_2) \Delta N_1 - \delta_{2k} d_k(\omega_2) \left(\bar{N}_k - \bar{N}_{k-} \right) - \int_{(k,1]} \delta_{2j} d_j(\omega_2) d\bar{N}_j \right] \\
= \;& c_1(\omega_1) + \frac{\delta_{1k} d_k(\omega_1)}{\delta_{2k} d_k(\omega_2)} \left[\underbrace{\int_{[0,1]} \delta_{2j} d_j(\omega_2) d\hat{N}_{2j}}_{\leq c_2(\omega_2)} - \underbrace{\left(\delta_{2k} d_k(\omega_2) \Delta N_2 + \int_{(k,1]} \delta_{2j} d_j(\omega_2) d\bar{N}_j \right)}_{= c_2(\omega_2)} \right] \leq c_1(\omega_1),
\end{aligned}$$

where: the second line follows by feasibility; $\bar{N} = \hat{N}_1 + \hat{N}_2$, the third line follows by rearranging and using the assumption that $(c, \hat{N})$ is incentive feasible; the fourth line follows by using the condition that (45) is strictly increasing; the fifth line by rearranging and using feasibility again; and the sixth line by our assumption that $(c, \hat{N})$ is incentive

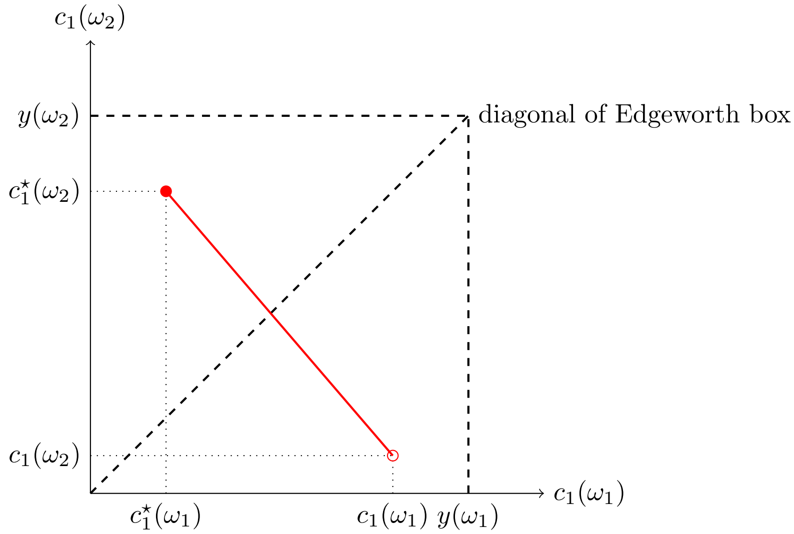

Figure 5: The Edgeworth box for the consumption of agent 1 in state ω_1 (x-axis) and in state ω_2 (y-axis).

feasible and by our observation that (21) must hold with equality given our choice of k and ΔN_2.

D.2 Proof of Lemma 3

Consider first the first-best allocation, $c^\star$. The first-order condition of the Planner's problem implies

$$\alpha_1 \left[c_1^\star(\omega)\right]^{-\gamma_1} - \alpha_2 \left[y(\omega) - c_1^\star(\omega)\right]^{-\gamma_2} = 0,$$

for all $\omega \in \Omega$. In terms of consumption share, $c(\omega)/y(\omega)$, this equation becomes:

$$\alpha_1 \left[\frac{c_1^\star(\omega)}{y(\omega)}\right]^{-\gamma_1} y(\omega)^{\gamma_2-\gamma_1} - \alpha_2 \left[1 - \frac{c_1^\star(\omega)}{y(\omega)}\right]^{-\gamma_2} = 0. \tag{48}$$

Since $\gamma_2 > \gamma_1$, this equation is strictly decreasing in the consumption share and strictly increasing in $y(\omega)$. Hence it follows that the consumption share is strictly increasing in $y(\omega)$, i.e., $c_1^\star(\omega_1)/y(\omega_1) < c_1^\star(\omega_2)/y(\omega_2)$. The inequality for $i = 2$ follows directly because consumption shares add up to one.

Now consider the equilibrium allocation, c. Assume, toward a contradiction, that $c_1(\omega_1)/y(\omega_1) \geq c_1(\omega_2)/y(\omega_2)$, i.e., the consumption shares of agent $i = 1$ lie below the diagonal of the Edgeworth box, as shown in Figure 5. Notice that since the first-best allocation, $c^\star$, satisfies the reverse inequality, it must lie strictly above the diagonal. This implies that $c^\star \neq c$. By strict concavity, social welfare evaluated at c is strictly smaller than social welfare evaluated at $c^\star$, and strictly smaller than social welfare at any point on the segment $(c, c^\star]$ linking c to $c^\star$, shown in red on the figure. Clearly, the segment $[c, c^\star)$ crosses the diagonal at some point $c^\dagger$, which may be c. Since $c^\dagger$ keeps the agent's consumption share constant across states, it can be made incentive feasible by giving agents the corresponding "proportional" tree allocation, i.e., a share in the market portfolio equal to their respective consumption share, $N_i^\dagger = c_i^\dagger(\omega_i)/y(\omega_i)\,\bar{N}$. But since $\delta < 1$, it follows that all incentive constraints are slack for $(c^\dagger, N^\dagger)$. Therefore, points on the segment $(c, c^\star]$ near $c^\dagger$ are incentive feasible as well. But they improve social welfare strictly relative to c, which is a contradiction.

D.3 Proof of Proposition 6

As for Lemma 2, we offer a proof in the general case when δ_{ij} is assumed to depend both on the identity of the tree holders and on the type of the tree, maintaining the restriction that

$$\frac{\delta_{1j} d_j(\omega_1)}{\delta_{2j} d_j(\omega_2)}, \tag{49}$$

is strictly increasing.

Given Lemma 2, what remains to be shown is that, for any incentive-feasible consumption allocation on the boundary, the distribution of trees is uniquely determined. We establish:

Lemma D.1 *Suppose that (20) and (21) holds with equality for some consumption allocation c, some $k \in [0,1]$ and some $(\Delta N_1, \Delta N_2) \geq 0$ such that $\Delta N_1 + \Delta N_2 = N_k - N_{k-}$. Then (c, N^+) is incentive feasible if and only if $N_1^+ = \Delta N_1 \mathbb{I}_{\{j=k\}} + \bar{N}\, \mathbb{I}_{\{j<k\}}$ and $N_2^+ = \Delta N_2 \mathbb{I}_{\{j=k\}} + \bar{N}\, \mathbb{I}_{\{j>k\}}$.*

The "if" part of Lemma D.1 follows because, with the proposed tree allocation, the incentive constraint of agent $i = 1$ binds in state ω_1, and that of agent $i = 2$ binds in state ω_2. It then follows that the two other incentive constraints are slack.

For the "only if" part, consider any tree allocation such that $(c, \hat{N}^+)$ is incentive feasible. Then the incentive constraint of agent $i = 1$ in state ω_1 writes:

$$c_1(\omega_1) = \int_{[0,k)} \delta_{1j} d_j(\omega_1)\, d\bar{N}_j + \delta_{1k} d_k(\omega_1) \Delta N_1 \geq \int \delta_{1j} d_j(\omega_1) d\hat{N}_{1j}^+$$

Using that $d\bar{N}_j = d\hat{N}_{1j}^+ + d\hat{N}_{2j}^+$ we then obtain that:

$$\int_{[0,k)} \delta_{1j} d_j(\omega_1)\, d\hat{N}_{2j}^+ + \delta_{1k} d_k(\omega_1) \Delta N_1 \geq \int_{(k,1]} \delta_{1j} d_j(\omega_1) d\hat{N}_{1j}^+ + \delta_{1k} d_k(\omega_1) \Delta \hat{N}_1, \tag{50}$$

where $\Delta \hat{N}_1 \equiv \hat{N}_{1k} - \hat{N}_{1k-}$. Proceeding analogously with the incentive constraint of agent $i = 2$ in state ω_2, we obtain:

$$\int_{(k,1]} \delta_{2j} d_j(\omega_2)\, d\hat{N}_{1j}^+ + \delta_{2k} d_k(\omega_2) \Delta N_2 \geq \int_{[0,k)} \delta_{2j} d_j(\omega_2) d\hat{N}_{2j}^+ + \delta_{2k} d_k(\omega_2) \Delta \hat{N}_2, \tag{51}$$

where $\Delta \hat{N}_2 \equiv \hat{N}_{2k} - \hat{N}_{2k-}$ Now multiply equation (50) by $\delta_{2k} d_k(\omega_2)$, and equation (51) by $\delta_{1k} d_k(\omega_1)$ and add the two inequalities. The $j = k$ terms cancel each others out because, by feasibility, $\Delta N_1 + \Delta N_2 = \Delta \hat{N}_1 + \Delta \hat{N}_2$. We thus obtain:

$$\int_{[0,k)} \delta_{1j} d_j(\omega_1) \delta_{2k} d_j(\omega_2)\, d\hat{N}_{2j}^+ + \int_{(k,1]} \delta_{2j} d_j(\omega_2) \delta_{1k} d_k(\omega_1)\, d\hat{N}_{1j}^+ \geq \int_{(k,1]} \delta_{1j} d_j(\omega_1) \delta_{2k} d_k(\omega_2) d\hat{N}_{1j}^+ + \int_{[0,k)} \delta_{2j} d_j(\omega_2) \delta_{1k} d_k(\omega_1) d\hat{N}_{2j}^+.$$

After rearranging:

$$\int_{[0,k)} \left[\delta_{1j} d_j(\omega_1) \delta_{2k} d_j(\omega_2) - \delta_{2j} d_j(\omega_2) \delta_{1k} d_k(\omega_1)\right] d\hat{N}_{2j}^+ \geq \int_{(k,1]} \left[\delta_{1j} d_j(\omega_1) \delta_{2k} d_k(\omega_2) - \delta_{2j} d_j(\omega_2) \delta_{1k} d_k(\omega_1)\right] d\hat{N}_{1j}^+$$

But, by (49), the integrand on the left-hand side is strictly negative over $[0, k)$, while the integrand on the right-hand side is strictly positive over $(k, 1]$. Therefore, both integrals are zero: agent $i = 2$ holds no tree in $[0, k)$ and all trees in $(k, 1]$, while agent $i = 1$ holds all trees in $[0, k)$ and no tree in $(k, 1]$. Plugging this back into the incentive compatibility constraint, we can determine each agent's holdings of tree k. Indeed, we obtain:

$$\delta_{1k}d_k(\omega_1)\Delta N_1 \geq \delta_{1k}d_k(\omega_1)\Delta\hat{N}_1 \text{ and } \delta_{2k}d_k(\omega_2)\Delta N_2 \geq \delta_{2k}d_k(\omega_1)\Delta\hat{N}_2.$$

Since $\Delta N_1 + \Delta N_2 = \Delta\hat{N}_1 + \Delta\hat{N}_2$, it follows that $\Delta N_1 = \Delta\hat{N}_1$ and $\Delta N_2 = \Delta\hat{N}_2$.

D.4 Proof of Proposition 7

With two agents, the zero-transfer equation (41) writes:

$$\bar{n}_2\alpha_1\mathbb{E}\left\{u_1'\left[c_1(\omega)\right]c_1(\omega)\right\} = \bar{n}_1\alpha_2\mathbb{E}\left\{u_2'\left[c_2(\omega)\right]c_2(\omega)\right\}$$

With CRRA utility, this can be simplified further:

$$\bar{n}_2\alpha_1\mathbb{E}\left[c_1(\omega)^{1-\gamma_1}\right] = \bar{n}_1\alpha_2\mathbb{E}\left[c_2(\omega)^{1-\gamma_2}\right],$$

so that:

$$\frac{\bar{n}_1}{\bar{n}_2} = \frac{\alpha_1\mathbb{E}\left[c_1(\omega)^{1-\gamma_1}\right]}{\alpha_2\mathbb{E}\left[c_2(\omega)^{1-\gamma_2}\right]}.$$

Now notice that, as α_1/α_2 increases, the solution of the Planner's problem moves to the northeast of the incentive-constrained Pareto set (see the proof of Proposition C.8). Clearly, this implies a strictly increasing relationship between $\bar{n}_1/\bar{n}_2$ and α_1/α_2.

D.5 Proof of Proposition 8

Suppose c is incentive feasible under tree distribution $\bar{N}$ and is such that the consumption share of agent $i = 1$ is higher in state ω_1 than in state ω_2, as is the case in equilibrium (the opposite case is symmetric). Lemma 2 implies that there exist $k \in [0,1]$ and $(\Delta N_1, \Delta N_2) \geq 0$, $\Delta N_1 + \Delta N_2 = \bar{N}_k - \bar{N}_{k-}$ such that the incentive compatibility constraints (20) and (21) are satisfied.

To establish the claim of the proposition, that c is incentive feasible under tree distribution $\bar{N}^*$, we need to find some $k^* \in [0,1]$ and $(\Delta N_1^*, \Delta N_2^*) \geq 0$, $\Delta N_1^* + \Delta N_2^* = \bar{N}_{k^*} - \bar{N}_{k^*-}$ such that the analog of conditions (20) and (21) for the new tree allocation (described with variables with a "$*$" subscript) are satisfied. It is thus sufficient to find some k^* and $(\Delta N_1^*, \Delta N_2^*) \geq 0$, $\Delta N_1^* + \Delta N_2^* = \bar{N}_{k^*} - \bar{N}_{k^*-}$ such that:

$$\int_{j\in[0,k^*)} jd\bar{N}_j^* + k^*\Delta N_1^* \leq \int_{j\in[0,k)} jd\bar{N}_j + k\Delta N_1 \tag{52}$$

and

$$\int_{j\in(k^*,1]} (1-j)d\bar{N}_j^* + (1-k^*)\Delta N_2^* \leq \int_{j\in(k,1]} (1-j)d\bar{N}_j + (1-k)\Delta N_2. \tag{53}$$

Let $k^* \in [0,1]$ and $(\Delta N_1^*, \Delta N_2^*) \geq 0$, $\Delta N_1^* + \Delta N_2^* = \bar{N}_{k^*} - \bar{N}_{k^*-}$, such that

$$\bar{N}_{k^*-}^* + \Delta N_1^* = \bar{N}_{k-} + \Delta N_1. \tag{54}$$

First note that (54) together with (23) and (24) imply that (52) holds if and only if (53) holds. Indeed, adding

$$\bar{N}_1^* - \left(\bar{N}_{k^*-}^* + \Delta N_1^*\right) - \int_{j\in[0,1]} jd\bar{N}_j^* = \bar{N}_1 - \left(\bar{N}_{k-} + \Delta N_1\right) - \int_{j\in[0,1]} jd\bar{N}_j$$

on each side of (52) yields (53).

Let us now show that (52) holds. Integrating the left-hand side of (52) by parts yields

$$\int_{j\in[0,k^*)} jd\bar{N}_j^* + k^*\Delta N_1^* = k^*\bar{N}_{k^*-}^* - \int_0^{k^*} \bar{N}_j^* dj + k^*\Delta N_1^*.$$

There are two cases to consider.

First case: $k^* \leq k$. Then, it follows from (54) that $\bar{N}_{k^*-}^* + \Delta N_1^* = \bar{N}_{k-} + \Delta N_1 \geq \bar{N}_{k-} \geq \bar{N}_j$ for $j \in (k^*, k)$. Thus,

$$\begin{aligned}
\int_{j\in[0,k^*)} jd\bar{N}_j^* + k^*\Delta N_1^* &\leq k^*\bar{N}_{k^*-} - \int_0^{k^*} \bar{N}_j^* dj + k^*\Delta N_1^* + \int_{k^*}^{k} (\bar{N}_{k^*-}^* + \Delta N_1^* - \bar{N}_j)dj \\
&= k\left(\bar{N}_{k^*-} + \Delta N_1^*\right) - \int_0^{k^*} \bar{N}_j^* dj - \int_{k^*}^{k} \bar{N}_j dj \\
&= k\left(\bar{N}_{k-} + \Delta N_1\right) - \int_0^{k^*} \bar{N}_j^* dj - \int_{k^*}^{k} \bar{N}_j dj \\
&\leq k\left(\bar{N}_{k-} + \Delta N_1\right) - \int_0^{k^*} \bar{N}_j dj - \int_{k^*}^{k} \bar{N}_j dj \\
&= k\left(\bar{N}_{k-} + \Delta N_1\right) - \int_0^{k} \bar{N}_j dj
\end{aligned}$$

where the second line is obtained by calculation, the third line follows from (54), the fourth line follows from the definition of second order stochastic dominance, the fifth line is obtained by calculation and is equal to the right-hand side of (52) after integration by parts.

Second case: $k < k^*$. Then

$$\begin{aligned}
\int_{j\in[0,k^*)} jd\bar{N}_j^* + k^*\Delta N_1^* &= k^*\left(\bar{N}_{k-} + \Delta N_1\right) - \int_0^{k^*} \bar{N}_j^* dj \\
&\leq k^*\left(\bar{N}_{k-} + \Delta N_1\right) - \int_0^{k^*} \bar{N}_j dj \\
&\leq k^*\left(\bar{N}_{k-} + \Delta N_1\right) - \int_0^{k} \bar{N}_j dj - \int_{k}^{k^*} \bar{N}_k dj \\
&\leq k^*\left(\bar{N}_{k-} + \Delta N_1\right) - \int_0^{k} \bar{N}_j dj - (k^* - k)\left(\bar{N}_{k-} + \Delta N_1\right) \\
&= k\left(\bar{N}_{k-} + \Delta N_1\right) - \int_0^{k} \bar{N}_j dj
\end{aligned}$$

where the first line follows by integration by part of (54), the second line follows from the definition of second order stochastic dominance, the third line follows from $\bar{N}$ being increasing, the fourth line follows from $\Delta N_2 \geq 0$, the fifth line is obtained by calculation and is equal to the right-hand side of (52) after integration by parts.

D.6 Proof of Proposition 10

Notice that, since the function ϕ_ℓ is the same for both agents, we have that $\delta_{1j}d_j(\omega_1)/\delta_{2j}d_j(\omega_2) = d_j(\omega_1)/d_j(\omega_2)$ is strictly increasing, so all our results apply.

The equilibrium is uniquely pinned down by a two-equation-in-two-unknown problem, for the ratio of the two budget constraints multipliers, $r = \frac{\lambda_1}{\lambda_2}$, and the threshold k determining tree ownership. To obtain the first equation, first note that the continuity of $j \mapsto (\delta_{1j}d_j(\omega_1))/(\delta_{2j}d_j(\omega_2))$ implies that for the threshold tree, the first-order condition with respect to tree holdings holds with an equality for both agents. Thus:

$$F(r,k) \equiv \mu_1(\omega_1)\delta_{1k}d_k(\omega_1) - r\mu_2(\omega_2)\delta_{2k}d_k(\omega_2) = 0. \tag{55}$$

where, from the first-order conditions we have that

$$\begin{aligned}
\mu_1(\omega_1) &= r\pi(\omega_1)u_2'\left[\int_0^1 \left(1-\delta_{1j}\mathbb{I}_{\{j<k\}}\right)d_j(\omega_1)\,d\bar{N}_j\right] - \pi(\omega_1)u_1'\left[\int_0^1 \delta_{1j}\mathbb{I}_{\{j<k\}}d_j(\omega_1)\,d\bar{N}_j\right] \\
\mu_2(\omega_2) &= \frac{1}{r}\pi(\omega_2)u_1'\left[\int_0^1 \left(1-\delta_{2j}\mathbb{I}_{\{j\geq k\}}\right)d_j(\omega_2)\,d\bar{N}_j\right] - \pi(\omega_2)u_2'\left[\int_0^1 \delta_{2j}\mathbb{I}_{\{j\geq k\}}d_j(\omega_2)\,d\bar{N}_j\right].
\end{aligned}$$

Notice that the continuity of the distribution of tree supplies mean that there is no atom, hence $\Delta N_1 = \Delta N_2 = 0$, i.e., the allocation of the supply of threshold tree between agents is irrelevant. The second equilibrium equation is (41) which here takes the form:

$$G(r,k) \equiv \mathbb{E}[u_1'(c_1(\omega))c_1(\omega)] - r\frac{\bar{n}_1}{\bar{n}_2}\mathbb{E}[u_2'(c_2(\omega))c_2(\omega)] = 0, \tag{56}$$

where $c_1(\omega_1) = \int_0^k \delta_{1j}d_j(\omega_1)\,d\bar{N}_j$, $c_2(\omega_1) = \int_0^1 d_j(\omega_1)\,d\bar{N}_j - c_1(\omega_1)$, $c_2(\omega_2) = \int_k^1 \delta_{2j}d_j(\omega_2)\,d\bar{N}_j$, and $c_1(\omega_2) = \int_0^1 d_j(\omega_2)\,d\bar{N}_j - c_2(\omega_2)$.

The function $F(r,k)/(\delta_{2k}d_k(\omega_2))$ is strictly increasing and continuous in both r and k. Moreover, one can explicitly solve for r as a function of k, $\rho(k)$. This function is strictly decreasing and, because of the Inada condition $u_i'(0) = +\infty$, goes to infinity as k goes to zero, $\lim_{k\to 0}\rho(k) = \infty$, and goes to zero as k goes to one, $\lim_{k\to 1}\rho(k) = 0$.

Since $\bar{N}_j$ is strictly increasing, it follows that both $c_1(\omega_1)$ and $c_1(\omega_2)$ are strictly increasing in k while both $c_2(\omega_1)$ and $c_2(\omega_2)$ are strictly decreasing in k. Recall that the coefficient of relative risk aversion are both less than one, $0 \leq \gamma_1 < \gamma_2 \leq 1$. Therefore, the function $G(r,k)$ is strictly decreasing in r and strictly increasing in k. Plugging in the function $\rho(k)$ defined above, we obtain a strictly increasing function $k \mapsto G(\rho(k),k)$. Given our earlier observation that $\lim_{k\to 0}\rho(k) = \infty$ and $\lim_{k\to 1}\rho(k) = 0$, it follows that $k \mapsto G(\rho(k),k)$ is strictly negative when $k \simeq 0$, and strictly positive when $k \simeq 1$. Thus, the equilibrium threshold is the unique solution of $G(\rho(k),k) = 0$. Clearly $c_1(\omega_1)$ increases with ε, while $c_2(\omega_2)$ stays the same. This implies that $\rho(k)$ shifts down, and that $G(\rho(k),k)$ shifts down as well. Hence $k(\varepsilon') < k(\varepsilon)$ if $\varepsilon' > \varepsilon$.

$$\frac{dk}{d\varepsilon} < 0.$$

E Supplementary appendix

E.1 Proof of Lemma C.1

For this proof, in order to apply some of the results in Chapter 12 of Stokey and Lucas (1989), we extend measures $M^+ \in \mathcal{M}_+$ to the entire real line, $\mathbb{R}$, by setting $M_j^+ = 0$ for all $j < 0$, and $M_j^+ = M_1^+$ for all $j \geq 1$. Now consider a sequence (c^k, N^{k+}), $k \in \mathbb{N}$, of incentive feasible allocations. Given that c^k belongs to a finite dimensional space and is bounded, it has a converging subsequence. Given that $\sum_{i \in I} N_i^{k+} = \bar{N}$, N_{ij}^+ is bounded above by $\bar{N}_j$ for all $(i,j) \in I \times \mathbb{R}$, an an application of Helly's selection Theorem (Theorem 12.9 in Stokey and Lucas (1989) easily extended to finite measures instead of distributions) shows that for each $i \in I$, N_i^{k+} has a subsequence such that $N_i^{\ell+}$ converging weakly in $\mathcal{M}_+$. Taken together, this means that there exists a subsequence $(c^\ell, N^{\ell+})$ of (c^k, N^{k+}) and some (c, N^+) such that $c^\ell \to c$ and $N_i^{\ell+} \Rightarrow N_i^+$ for each $i \in I$.

What is left to show is that (c, N^+) is incentive feasible. Given that $j \mapsto d_j(\omega)$ and $j \mapsto \delta_{ij}$ are continuous, the definition of weak convergence allows us to assert that, since the feasibility constraint for consumption, (9), and the incentive compatibility constraints, (38), hold for each $(c^\ell, N^{\ell+})$, then it must also hold in the limit for (c, N^+). The only difficulty is to show that the feasibility constraint for holdings is also satisfied. For this we rely on the characterization of weak convergence provided by Theorem 12.8 in Stokey and Lucas (1989), easily extended to finite measures. It asserts that $N_i^{\ell+}$ converges pointwise at each continuity point of their limit, N_i^+. Therefore, for any $j \in \mathbb{R}$ such that all $(N_i^+)_{i \in I}$ are continuous, we have:

$$\sum_{i \in I} N_{ij}^{\ell+} \to \sum_{i \in I} N_{ij}^+.$$

But recall that the feasibility constraint for holdings is satisfied for each j: $\sum_{i \in I} N_{ij}^{\ell+} = \bar{N}_j$. Together with the above, this implies that

$$\sum_{i \in I} N_{ij}^+ = \bar{N}_j,$$

for all $j \in \mathbb{R}$ such as all $(N_i^+)_{i \in I}$ are continuous. Now recall that the N_i^+'s are increasing functions, and so have countably many discontinuity points. This implies that for any $j \in \mathbb{R}$, there is a sequence of $j_n \downarrow j$ such that j_n is a continuity point for all $(N_i^+)_{i \in I}$. Hence, for all j_n, we have

$$\sum_{i \in I} N_{ij_n}^+ = \bar{N}_{j_n}^+.$$

Since $j \mapsto N_{ij}^+$ and $\bar{N}_j$ are all right continuous functions we can take the limit $j_n \downarrow j$ and obtain that $\sum_{i \in I} N_{ij}^+ = \bar{N}_j$ for all $j \in \mathbb{R}$, as required.

E.2 Proof of Proposition C.1

In all what follow we let:

$$y(\omega) \equiv \int d_j(\omega)\, d\bar{N}_j, \; \underline{y} \equiv \min_{\omega \in \Omega} y(\omega), \text{ and } \bar{y} \equiv \max_{\omega \in \Omega} y(\omega).$$

Proof that $\Gamma^\star(\alpha)$ is not empty. We first show that the supremum is achieved. The only difficulty with this proof arises when $\alpha_i > 0$ and $u_i(0) = -\infty$ for some $i \in I$, because in this case the objective is unbounded as $c_i(\omega) \to 0$. However, in the planner's problem, one can restrict attention to $c_i(\omega)$ that are bounded away from zero. To see this, we first note that $c_h(\omega) = y(\omega)/I$, $N_h + = \bar{N}/I$, for all $h \in I$, is an incentive-feasible allocation, implying that:

$$W^\star(\alpha) \geq \sum_{i \in I} \alpha_i \sum_{\omega \in \Omega} \pi(\omega) u_i \left[y(\omega)/I \right] \geq \underline{W}_i(\alpha_i) \text{ where } \underline{W}_i(\alpha_i) \equiv \alpha_i \pi(\omega) u_i(\underline{y}/I) + \sum_{h \neq i} \min\left\{ u_h\left(\underline{y}/I\right), 0 \right\}.$$

Also, for any allocation (c, N^+), we have that

$$W(\alpha, c, N^+) \leq \alpha_i \pi(\omega) u_i \left[c_i(\omega) \right] + \sum_{h \neq i} \max\{u_h(\bar{y}), 0\},$$

for any $i \in I$. Now consider the equation

$$\alpha_i \pi(\omega) u_i \left[c_i(\omega)\right] + \sum_{h \neq i} \max\{u_k(\bar{y}), 0\} = \underline{W}_i(\alpha_i, \omega) = \alpha_i \pi(\omega) u_i(\underline{y}/I) + \sum_{h \neq i} \min\left\{u_k\left(\underline{y}/I\right), 0\right\},$$

for some fixed i such that $\alpha_i > 0$ and $u_i(0) = -\infty$. Since $u_i(0) = -\infty$, the left-hand side is smaller than the right-hand side when $c \to 0$. Since $u_i(c)$ is strictly increasing, the left-hand side is greater than the right-hand side when $c > \underline{y}/I$, it follows that the equation has a unique solution, which is less than $\underline{y}/I$. Clearly, the solution is increasing and continuous in α_i. Let $\underline{c}_i(\alpha_i)$ be half of the minimum of these solutions across all $\omega \in \Omega$. By construction, for any allocation (c, N^+) such that $c_i(\omega) < \underline{c}_i(\alpha_i)$ for some $\omega \in \Omega$, $W(\alpha, c, N^+)$ is strictly less than the value attained by $c_h(\omega) = y(\omega)/I$ and $N_h^+ = \bar{N}/I$, and so cannot be optimal. If we let $\underline{c}_i(\alpha_i) = 0$ for other i, that is for $i \in I$ such that $\alpha_i = 0$ or $u_i(0) = 0$, then, in the Planner's problem, one can restrict attention to allocation such that $c_i(\omega) \geq \underline{c}_i(\alpha_i)$, which we write as $c \geq \underline{c}(\alpha)$. Notice that, by construction, the objective of the planner is continuous over $c \geq \underline{c}(\alpha)$.

Now to show that there is a solution consider any sequence (c^k, N^{k+}) of incentive-feasible allocation such that $W(\alpha, c^k, N^{k+}) \to W^\star(\alpha)$. From the above remark we can focus on a sequence such that $c^k \geq \underline{c}(\alpha)$. Now, by Lemma C.1, there exists some incentive feasible allocation (c, N^+) and a subsequence $(c^\ell, N^{\ell+})$ such that $c^\ell \to c$ and $N^{\ell+} \to N^+$. Going to the limit in the Planner's objective, we obtain that $W(\alpha, c, N^+) = W^\star(\alpha)$.

Proof that $\Gamma^\star(\alpha)$ is weakly compact. The argument is the same as in the last paragraph, except that we now consider a sequence $(c^k, N^{k+}) \in \Gamma^\star(\alpha)$.

Proof that $\Gamma^\star(\alpha)$ convex-valued. This follows because the objective is concave and the constraints linear.

Proof that $W^\star(\alpha)$ is continuous and $\Gamma^\star(\alpha)$ has a weakly closed graph. Consider any $\bar{\alpha} \geq 0$ such that $\sum_{i \in I} \bar{\alpha}_i = 1$ and any sequence $\alpha^k \to \bar{\alpha}$ and an associated sequence $(c^k, N^{k+}) \in \Gamma^\star(\alpha^k)$. Without loss of generality for this proof, assume that $W^\star(\alpha^k)$ converges to some limit,[25] and that (c^k, N^{k+}) converges weakly towards some incentive feasible allocation (c, N^+).[26] We want to show that $W^\star(\alpha^k) \to W^\star(\alpha)$ and that $(c, N^+) \in \Gamma^\star(\alpha)$. Let $I_0 = \{i \in I \,:\, \bar{\alpha}_i = 0 \text{ and } u_i(0) = -\infty\}$. We have:

$$W^\star(\alpha^k) = \sum_{i \notin I_0} \alpha_i^k \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i^k(\omega)\right] + \sum_{i \in I_0} \alpha_i^k \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i^k(\omega)\right]. \tag{57}$$

By our maintained assumptions, both the left-hand side and the first term on the right-hand side have a limit as $k \to \infty$. Hence, the second term on the right-hand side has a limit as well. We argue that this limit must be negative. Indeed, for $i \in I_0$, if $\lim c_i^k(\omega) > 0$, then $\lim \alpha_i^k u_i \left[c_i^k(\omega)\right] = 0$. If $\lim c_i^k(\omega) = 0$, then $\alpha_i^k u_i \left[c_i^k(\omega)\right] \leq 0$ for k large enough. Hence,

$$\lim \sum_{i \in I_0} \alpha_i^k \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i^k(\omega)\right] \leq 0.$$

Therefore:

$$\lim W^\star(\alpha^k) \leq \sum_{i \notin I_0} \bar{\alpha}_i \sum_{\omega \in \Omega} \pi(\omega) u_i \left[\lim c_i^k(\omega)\right] \leq W^\star(\bar{\alpha}), \tag{58}$$

since $(\lim c^k, \lim N^k)$ is incentive feasible.

To show the reverse inequality, for all $i \in I_0$, choose some $\phi_i > 0$ such that $1 + \phi_i(1 - \gamma_i) > 0$, where $\gamma_i > 1$ is the assumed CRRA bound for $u_i(c)$. Let $\beta(\alpha) \equiv \sum_{i \in I_0} (\alpha_i)^{\phi_i}$. Since $\lim \alpha_i^k = 0$ for all $i \in I_0$, we have that $\lim \beta(\alpha^k) = 0$, hence $\beta(\alpha^k) < 1$ for all k large enough. Now take any $(c, N^+) \in \Gamma^\star(\bar{\alpha})$. For all $i \in I_0$ we have that $\bar{\alpha}_i = 0$, which clearly implies that $c_i(\omega) = N_i^+ = 0$, i.e., $i \notin I_0$ consume the aggregate endowment and holds the

[25] Indeed, since $W^\star(\alpha)$ is bounded below by $\min\{\underline{W}_i(\alpha_i, \omega), i \in I, \alpha_i \in [0,1], \omega \in \Omega\}$ and is clearly bounded above, to show convergence towards $W^\star(\alpha)$ it is sufficient to show that every convergent subsequence of $W^\star(\alpha^k)$ converges towards $W^\star(\alpha)$.

[26] From Lemma C.1, we can always find a convergence subsequence with this property.

aggregate tree supplies. Therefore, if we scale down the consumption and tree holding of $i \notin I_0$ by $1-\beta(\alpha^k)$, we keep the allocation of $i \notin I_0$ incentive compatible and we free up $\beta(\alpha^k)y(\omega)$ consumption, and $\beta(\alpha^k)\bar{N}$ trees. We can then re-distribute this consumption and these trees by giving to each agent $i \in I_0$ a consumption equal to $y(\omega)\left(\alpha_i^k\right)^{\phi_i}$ and a tree allocation equal to a fraction $\left(\alpha_i^k\right)^{\phi_i}$ of the market portfolio, $\bar{N}$. Because the consumption of $i \in I_0$ is proportional to its portfolio payoff, the allocation of $i \in I_0$ is incentive compatible. Therefore, this process of scaling down the consumption of $i \notin I_0$ and redistributing to $i \in I_0$, leads to an incentive feasible allocation. Hence, we have that:

$$W^\star(\alpha^k) \geq \sum_{i \notin I_0} \alpha_i^k \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i(\omega)(1-\beta(\alpha^k))\right] + \sum_{i \in I_0} \alpha_i^k \sum_{\omega \in \Omega} \pi(\omega) u_i \left[y(\omega)\left(\alpha_i^k\right)^{\phi_i}\right].$$

The first term converges to $W^\star(\bar{\alpha})$. Using the assumed CRRA bound, $0 < |u(c)| < |K| c^{1-\gamma_i}$ for c close to zero, one sees that the second term goes to zero: indeed $\alpha_i^k \left| u_i\left[y(\omega)\left(\alpha_i^k\right)^{\phi_i}\right]\right|$ is bounded above by $|K| y(\omega)^{1-\gamma_i}\left(\alpha_i^k\right)^{1+(1-\gamma_i)\phi_i}$, which goes to zero since $\lim \alpha_i^k = 0$ and ϕ_i was chosen so that $1+\phi_i(1-\gamma_i) > 0$. Hence, we obtain that $\lim W^\star(\alpha^k) \geq W^\star(\bar{\alpha})$. Taken together we have that

$$\lim W^\star(\alpha^k) \geq \sum_{i \notin I_0} \bar{\alpha}_i \sum_{\omega \in \Omega} \pi(\omega) u_i \left[\lim c_i^k(\omega)\right] = W^\star(\bar{\alpha}). \tag{59}$$

Taken together, (58) and (59) imply that

$$\lim W^\star(\alpha^k) = \sum_{i \notin I_0} \alpha_i \sum_{\omega \in \Omega} u_i \left[\lim c_i^k(\omega)\right] = W(\bar{\alpha}) \text{ and } \lim \sum_{i \in I_0} \alpha_i^k \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i^k(\omega)\right] = 0.$$

This establishes that $W^\star(\alpha)$ is continuous and that $\Gamma^\star(\alpha)$ has a closed graph.

Proof that $\lim \alpha_i^k u'\left[c_i^k(\omega)\right] c_i^k(\omega) = 0$ if $\lim \alpha_i^k = 0$. Consider any sequence $\alpha^k \to \bar{\alpha}$ and any associated sequence (not necessarily converging) (c^k, N^{k+}) in $\Gamma^\star(\alpha^k)$. Since we have shown that $\Gamma^\star(\alpha)$ has a weakly closed graph, it follows that any converging subsequence of (c^k, N^{k+}) has a limit belonging to $\Gamma^\star(\bar{\alpha})$. Since the Planner finds optimal to give zero consumption to agents with zero weight, it follows that $\lim c_i^k(\omega) = 0$ for all i such that $\bar{\alpha}_i = 0$.

If $u_i(0) = 0$, then the result that $\lim \alpha_i^k u'\left[c_i^k(\omega)\right] c_i^k(\omega) = 0$ follows from the inequality $0 \leq u_i'(c)c \leq u_i(c)$. If $u_i(0) = -\infty$, we need a different argument. Write $W^\star(\alpha^k) = W_1^k + W_2^k$, where

$$W_1^k \equiv \sum_{i \notin I_0} \alpha_i \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i^k(\omega)\right] \text{ and } W_2^k \equiv \sum_{i \in I_0} \alpha_i \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i^k(\omega)\right].$$

By assumption, we have that $\lim \left(W_1^k + W_2^k\right) = W^\star(\bar{\alpha})$. Notice that W_1^k is bounded. Indeed, it is clearly bounded above because the constraint set is bounded. It is bounded below because, for any $i \notin I_0$ such that $u_i(0) = -\infty$, $\bar{\alpha}_i > 0$ and so α_i^k and hence $\underline{c}_i(\alpha_i^k)$ is bounded away from zero for k large enough. Given boundedness, we can extract some convergent subsequence W_1^ℓ of W_1^k. Since consumption and tree holdings are incentive feasible, it follows from Lemma C.1 that there exists a weakly convergent subsequence (c^p, N^{p+}) of $(c^\ell, N^{\ell+})$. Clearly, $\lim W_1^p = \lim W_1^\ell$. But, using the results of the previous paragraph, we have that $\lim W_1^p = W^\star(\bar{\alpha})$. Hence all convergent subsequences of W_1^k have the same limit $W^\star(\bar{\alpha})$, implying that $\lim W_1^k = W^\star(\bar{\alpha})$ and that $\lim W_2^k = 0$. It follows that, asymptotically as $k \to \infty$, the aggregate consumption of agents $i \notin I_0$ is arbitrarily close to $y(\omega)$, and the consumption of each agent $i \in I_0$ is arbitrarily close to zero. Therefore, for all k large enough, all terms in W_2^k are negative. Hence, for k large enough, we that for all $i \in I_0$, $W_2^k \leq \alpha_i^k \pi(\omega) u_i\left[c_i^k(\omega)\right] \leq 0$. Since $\lim W_2^k = 0$, it follows that $\lim \alpha_i^k \pi(\omega) u_i\left[c_i^k(\omega)\right] = 0$ as well. The result then follows from the CRRA bound $0 \leq u_i'(c)c \leq \gamma_i |u_i(c)|$.

E.3 Proof of Proposition C.2

Fix any feasible $N^+ \in \mathcal{M}_+$ and let:

$$W(\alpha \,|\, N^+) = \max \sum_{i \in I} \alpha_i U_i(c_i)$$

with respect to $c \in \mathbb{R}_+^{|\Omega| \times |I|}$, and subject to

$$\sum_{i \in I} c_i(\omega) \le \sum_{i \in I} \int d_j(\omega) dN^+_{ij} \quad \forall \omega \in \Omega$$
$$c_i(\omega) \ge \int \delta_{ij} d_j(\omega) dN^+_{ij} \quad \forall (i,\omega) \in I \times \Omega.$$

From Corollary 28.3 in Rockafellar (1970), $c \in \mathbb{R}_+^{|\Omega| \times |I|}$ is an optimal solution only if there exists multipliers $\hat{q} \in \mathbb{R}_+^{|\Omega|}$ and $\hat{\mu} \in \mathbb{R}_+^{|\Omega| \times |I|}$ such that:

$$\begin{aligned}
\alpha_i \frac{\partial U_i}{\partial c_i(\omega)} + \hat{\mu}_i(\omega) &\le \hat{q}(\omega) \\
\hat{q}(\omega) \left[\sum_{i \in I} \int d_j(\omega)\, dN^+_{ij} - \sum_{i \in I} c_i(\omega) \right] &= 0, \quad \forall \omega \in \Omega \\
\hat{\mu}_i(\omega) \left[c_i(\omega) - \int \delta_{ij} d_j(\omega)\, dN^+_{ij} \right] &= 0, \quad \forall (i,\omega) \in I \times \Omega.
\end{aligned}$$

Notice that there exists multipliers such that the top first-order condition, with respect to $c_i(\omega)$, holds with equality. Indeed, if it holds with a strict inequality for some $\hat{\mu}_i(\omega)$ and some (i,ω), then $c_i(\omega) = 0$ and so the incentive constraint holds with equality. So increasing $\hat{\mu}_i(\omega)$ leaves the complementary slackness conditions unchanged.

Now consider any other feasible $\hat{N}^+ \in \mathcal{M}_+$. Clearly, for any $h \in [0,1]$, $(1-h)N^+ + h\hat{N}^+ = N^+ + h(\hat{N}^+ - N^+)$ is also feasible. In the optimization problem associated with $W\left(\alpha \,|\, N^+ + h\left[\hat{N}^+ - N^+\right]\right)$, we take the derivative of the Lagrangian with respect to h, and we evaluate this derivative at $h = 0$, given some optimal consumption for $W(\alpha \,|\, N^+)$ and Lagrange multipliers satisfying the first order-conditions. We obtain:

$$\begin{aligned}
L_h &= \sum_{i \in I} \int \left[\sum_{\omega \in \Omega} \hat{q}(\omega) d_j(\omega) - \sum_{\omega \in \Omega} \hat{\mu}_i(\omega) \delta_{ij} d_j(\omega) \right] \left[d\hat{N}^+_{ij} - dN^+_{ij} \right] \\
&= \sum_{i \in I} \int \hat{v}_{ij} \left[d\hat{N}^+_{ij} - dN^+_{ij} \right],
\end{aligned}$$

where, for any set of Lagrange multipliers, $\hat{v}_{ij} \equiv \sum_{\omega \in \Omega} \hat{q}(\omega) d_j(\omega) - \sum_{\omega \in \Omega} \hat{\mu}_i(\omega) \delta_{ij} d_j(\omega)$. Notice that $\hat{q}(\omega)$ is uniquely determined by the first-order conditions[27] but $\hat{\mu}_i(\omega)$ may not, when $c_i(\omega) = 0$. One easily sees in particular that any

$$0 \le \hat{\mu}_i(\omega) \le \hat{q}(\omega) - \alpha_i \frac{\partial U_i}{\partial c_i(\omega)}$$

solves the first-order conditions. Let $\hat{V}$ denote the set of $\hat{v}_{ij}$ that is generated by all $\hat{q}$ and $hat\mu_i(\omega)$ solving the first-order conditions. It follows from Corollary 5 in Milgrom and Segal (2002) that the right-

[27] Indeed for any $\omega \in \Omega$, there exists some$i \in I$ such that the incentive compatibility constraint does not bind. For this $i \in I$, $c_i(\omega) > 0$ and so the first-order condition holds with equality. If $u_i(c)$ is linear, then $\alpha_i \partial U_i / \partial c_i(\omega) = \alpha_i$ is uniquely determined. If $u_i(c)$ is strictly concave, then $c_i(\omega)$ is uniquely determined and so is $\alpha_i \partial U_i / \partial c_i(\omega)$. Using the first-order condition, it then follows that $\hat{q}(\omega)$ is uniquely determined.

derivative of $W\left(\alpha \,|\, N^{+} + h\left[\hat{N}^{+} - N^{+}\right]\right)$ at $h = 0$ is

$$\frac{d}{dh} W\left(\alpha \,|\, N^{+} + h\left[\hat{N}^{+} - N^{+}\right]\right)\Bigg|_{h=0^{+}} = \min_{\hat{v} \in \hat{V}} \sum_{i \in I} \int \hat{v}_{ij} \left[d\hat{N}^{+}_{ij} - dN^{+}_{ij}\right].$$

Next we determine which $\hat{v} \in \hat{V}$ achieves the minimum above. We first notice that $\int \hat{v}_{ij} dN^{+}_{ij}$ does not depend on the particular choice of $\hat{v} \in \hat{V}$. Indeed, whenever $\hat{v}_{ij}$ is not uniquely determined, it is because $c_i(\omega) = 0$ for some $\omega \in \Omega$. But from the incentive compatibility constraint, it then follows that $\int \delta_{ij} d_j(\omega)\, dN^{+}_{ij} = 0$, so $\hat{\mu}_i(\omega) \int \delta_{ij} d_j(\omega)\, dN^{+}_{ij} = 0$ as well and $\hat{v}_{ij} = \sum_{\omega \in \Omega} \hat{q}(\omega) d_j(\omega)$, which is uniquely determined. Now, since $\hat{N}^{+}_{ij}$ is a positive measure, $\int \hat{v}_{ij} d\hat{N}^{+}_{ij}$ is minimized when $\hat{v}_{ij}$ is smallest, which occurs when $\hat{\mu}_i(\omega)$ is largest, that is, when it is chosen so that the first-order condition with respect to $c_i(\omega)$ holds with equality.

Taken together, we obtain that a necessary condition for a feasible N to be optimal is that:

$$\sum_{i \in I} \int \hat{v}_{ij} \left[d\hat{N}^{+}_{ij} - dN^{+}_{ij}\right] \leq 0, \tag{60}$$

for all feasible $\hat{N}^{+}$, where $\hat{v}_{ij} = \sum_{\omega \in \Omega} \hat{q}(\omega) d_j(\omega) - \sum_{\omega \in \Omega} \hat{\mu}_i(\omega) \delta_{ij} d_j(\omega)$ and $\hat{\mu}_i(\omega)$ is chosen so that the first-order condition with respect to $c_i(\omega)$ holds with equality. The proof is concluded by the following Lemma:

Lemma E.1 *Condition (60) holds if and only if* $\int \left[\max_{k \in I} \hat{v}_{kj} - \hat{v}_{ij}\right] dN^{+}_{ij} = 0$ *for all* $i \in I$.

For necessity, consider the correspondence $\Gamma(j) \equiv \arg\max_{k \in I} \hat{v}_{kj}$. By the Measurable Selection Theorem (Theorem 7.6 in Stokey and Lucas (1989)), there exists a measurable selection $\gamma(j)$. Consider then the tree allocation:

$$\hat{N}^{+}_{ij} = \int_0^j \mathbb{I}_{\{\gamma(k)=i\}} d\bar{N}_k,$$

which gives the supply of tree k to one agent with the highest valuation, $v_{\gamma(k)k}$. Condition (60) implies that:

$$\begin{aligned} 0 \geq \sum_{i \in I} \int \hat{v}_{ij} \left[d\hat{N}^{+}_{ij} - dN^{+}_{ij}\right] &= \sum_{i \in I} \int \hat{v}_{ij} \mathbb{I}_{\{\gamma(j)=i\}}\, d\bar{N}_j - \sum_{i \in I} \hat{v}_{ij}\, dN^{+}_{ij} \\ &= \int \max_{k \in I} \hat{v}_{kj} d\bar{N}_j - \sum_{i \in I} \int \hat{v}_{ij} dN^{+}_{ij} \\ &= \sum_{i \in I} \int \left(\max_{k \in I} \hat{v}_{kj} - \hat{v}_{ij}\right) dN^{+}_{ij}, \end{aligned}$$

where the second equality follows because $\sum_{i \in I} \hat{v}_{ij} \mathbb{I}_{\{\gamma(j)=i\}} = \max_{k \in I} \hat{v}_{kj}$, and the third equality follows because $\bar{N} = \sum_{i \in I} N_i$. But each term in the sum is positive since $\max \hat{v}_{kj} - \hat{v}_{ij} \geq 0$. It thus follows that each term in the sum is zero, and we are done.

For sufficiency, write

$$\begin{aligned} \sum_{i \in I} \int \hat{v}_{ij} \left[d\hat{N}^{+}_{ij} - dN^{+}_{ij}\right] &= \sum_{i \in I} \int \hat{v}_{ij}\, d\hat{N}^{+}_{ij} - \sum_{i \in I} \int \max_{k \in I} v_{kj} dN^{+}_{ij} \\ &= \sum_{i \in I} \int \hat{v}_{ij}\, d\hat{N}^{+}_{ij} - \int \max_{k \in I} v_{kj} d\bar{N}_j \\ &= \sum_{i \in I} \int \left[\hat{v}_{ij} - \max_{k \in I} v_{kj}\right] d\hat{N}^{+}_{ij} \leq 0. \end{aligned}$$

where the last equality follows because $\hat{N}^+$ is feasible, so $\bar{N}_j = \sum_{i \in I} \hat{N}^+_{ij}$.

E.4 Proof of Proposition C.3

Consider any (c, N^+) and multipliers $\hat{q}$, $\hat{\mu}$ and $\hat{p}$ satisfying the first-order conditions in the Proposition. Now let $(\hat{c}, \hat{N}^+)$ denote any other feasible allocation. We have:

$$\begin{aligned}
& \sum_{i\in I} \alpha_i U_i(c_i) - \sum_{i \in I} \alpha_i U_i(\hat{c}_i) \\
\geq \quad & \sum_{i \in I} \sum_{\omega \in \Omega} \frac{\partial U_i}{\partial c_i(\omega)} \left[c_i(\omega) - \hat{c}_i(\omega)\right] = \sum_{i \in I} \sum_{\omega \in \Omega} \left[\hat{q}(\omega) - \hat{\mu}_i(\omega)\right] \left[c_i(\omega) - \hat{c}_i(\omega)\right] \\
= \quad & \sum_{\omega \in \Omega} \hat{q}(\omega) \left[\sum_{i \in I} c_i(\omega) - \sum_{i \in I} \int d_j(\omega)\, dN^+_{ij}\right] - \sum_{\omega \in \Omega} \hat{q}(\omega) \left[\sum_{i \in I} \hat{c}_i(\omega) - \sum_{i \in I} \int d_j(\omega)\, d\hat{N}^+_{ij}\right] \\
& - \sum_{i \in I} \sum_{\omega \in \Omega} \hat{\mu}_i(\omega) \left[c_i(\omega) - \int \delta_{ij} d_j(\omega) dN^+_{ij}\right] + \sum_{i \in I} \sum_{\omega \in \Omega} \hat{\mu}_i(\omega) \left[\hat{c}_i(\omega) - \int \delta_{ij} d_j(\omega) d\hat{N}^+_{ij}\right] \\
& + \sum_{i \in I} \int \hat{v}_{ij} \left[dN^+_{ij} - d\hat{N}^+_{ij}\right] \geq \sum_{i \in I} \int \hat{v}_{ij} \left[dN^+_{ij} - d\hat{N}^+_{ij}\right],
\end{aligned}$$

where the last inequality follows from the complementarity slackness for (c, N^+), and from the feasibility of $(\hat{c}, \hat{N})$. Now since both N^+ and $\hat{N}^+$ are feasible, we have that:

$$\hat{p} \cdot \bar{N} = \hat{p} \cdot \sum_{i \in I} N_{ij} = \hat{p} \cdot \sum_{i \in I} \hat{N}_{ij}.$$

Hence, adding and subtracting $\hat{p} \cdot \bar{N}$, we obtain:

$$\sum_{i \in I} \int \hat{v}_{ij} \left[dN^+_{ij} - d\hat{N}^+_{ij}\right] = \sum_{i \in I} \left[\hat{p} \cdot \hat{N}^+_{ij} - \int \hat{v}_{ij}\, d\hat{N}_{ij}\right] - \sum_{i \in I} \left[\hat{p} \cdot N^+_{ij} - \int \hat{v}_{ij}\, dN^+_{ij}\right] \geq 0$$

where the last inequality follows from the first-order condition with respect to N^+.

E.5 Proof of Proposition C.5

Necessity. let (c, N^+) and (p, q) be an equilibrium. Since $\bar{n}_i > 0$, it follows from the first-order conditions of the agent's problem that $\lambda_i > 0$. By direct comparison of first-order conditions, one can then verify that the equilibrium allocation solves the Planner's Problem with weights

$$\alpha_i = \frac{1/\lambda_i}{\sum_{k \in I} 1/\lambda_k}.$$

The associated Lagrange multipliers are $\hat{\mu}_i(\omega) = \alpha_i \mu_i(\omega)$, $\hat{q}(\omega) = \beta q(\omega)$ and $\hat{v}_{ij} = \beta v_{ij}$ and $\hat{p} = \beta p$, where $\beta \equiv \left[\sum_{i \in I} 1/\lambda_i\right]^{-1}$. Finally, we have from Lemma C.2 that:

$$\alpha_i \sum_{\omega \in \Omega} \frac{\partial U_i}{\partial c_i(\omega)} c_i(\omega) = \bar{n}_i \int \hat{p}_j\, d\bar{N}_j.$$

Adding up across all $i \in I$ and using $\sum_{i \in I} \bar{n}_i = 1$ yields the desired condition.

Sufficiency. Consider any solution of the Planner's problem satisfying the conditions stated in the Proposition. Notice that the second condition implies that $\alpha_i > 0$. Using Proposition C.2 we obtain associated multipliers $\hat{q}$, $\hat{\mu}$ and $\hat{p}$. We do not assume here that $\hat{p}$ has a dot product representation: as in Proposition C.2 we only assume that $\hat{p}$ is a continuous linear functional. Consider then the candidate equilibrium prices $q(\omega) = \hat{q}(\omega)$ and $p = \hat{p}$, where the definition of equilibrium is extended in the obvious way when $\hat{p}$ does not have a dot product representation. Then, by direct comparison of first-order conditions, one sees that the component (c_i, N_i^+) of the Planner's allocation solves the necessary and sufficient conditions of agent $i \in I$'s problem, in Proposition C.4, except perhaps for the budget feasibility condition and the associated complementary slackness condition. The associated multipliers are $\lambda_i = 1/\alpha_i$, $\mu_i(\omega) = \hat{\mu}_i(\omega)/\alpha_i$ and $v_{ij} = \hat{v}_{ij}$. Therefore, to complete the proof, we need to verify that (c_i, N_i^+) satisfies budget feasibility. For this we calculate the gap between the left- and the right-hand sides of the budget constraint:

$$\begin{aligned}
T_i \quad &\equiv \quad \sum_{\omega \in \Omega} q(\omega) c_i(\omega) + p \cdot N_i^+ - \bar{n}_i p \cdot \bar{N} - \sum_{\omega \in \Omega} q(\omega) \int d_j(\omega)\, dN_{ij} \\
&= \quad \sum_{\omega \in \Omega} \left[\alpha_i \frac{\partial U_i}{\partial c_i(\omega)} + \hat{\mu}_i(\omega) \right] c_i(\omega) + \hat{p} \cdot N_i^+ - \bar{n}_i \hat{p} \cdot \bar{N} - \sum_{\omega \in \Omega} \hat{q}(\omega) \int d_j(\omega)\, dN_{ij} \\
&= \quad \sum_{\omega \in \Omega} \alpha_i \frac{\partial U_i}{\partial c_i(\omega)} c_i(\omega) - \bar{n}_i \hat{p} \cdot N + \hat{p} \cdot N_i^+ - \int \hat{v}_{ij} dN_{ij}^+ \\
&= \quad \sum_{\omega \in \Omega} \alpha_i \frac{\partial U_i}{\partial c_i(\omega)} c_i(\omega) - \bar{n}_i \hat{p} \cdot \bar{N}
\end{aligned}$$

where we substituted in the Planner's first-order and complementary-slackness conditions. Since aggregate ressource feasibility implies aggregate budget feasibility, it follows that $\sum_{i \in I} T_i = 0$. Since, in addition, $\sum_{i \in I} \bar{n}_i = 1$, we obtain that:

$$\hat{p} \cdot N = \sum_{k \in I} \sum_{\omega \in \Omega} \alpha_k \frac{\partial U_k}{\partial c_k(\omega)} c_k(\omega).$$

Since (c, N^+) satisfies the second condition stated in the Proposition, we obtain that $T_i = 0$, so budget balance holds.

E.6 Proof of Proposition C.6

Proof that $\Delta^\star(\alpha)$ is convex-valued. To show that $\Delta^\star(\alpha)$ is convex valued, we note that when $u_i(c)$ is strictly concave, $c_i(\omega)$ is uniquely determined, and so the term $\pi(\omega) u_i' \left[c_i(\omega)\right] c_i(\omega)$ is the same for all $(c, N^+) \in \Gamma^\star(\alpha)$. When $u_i(c)$ is linear, then $u'(c)c = c$ is linear. Taken together, this means that the function defining $\Delta^\star(\alpha)$ preserves the convexity of $\Gamma^\star(\alpha)$.

Proof that $\Delta^\star(\alpha)$ has a closed graph. Consider any converging sequence of α^k and $\Delta^k \in \Delta^\star(\alpha^k)$, generated by a sequence $(c^k, N^{k+}) \in \Gamma^\star(\alpha^k)$. Since $\Gamma^\star(\alpha^k)$ is included in the set of incentive feasible allocations, which by Lemma C.1 is weakly compact, we can extract a weakly convergent subsequence $(c^\ell, N^{\ell+})$ of (c^k, N^{k+}). Since we know from Proposition C.1 that $\Gamma^\star(\alpha)$ has a weakly closed graph, it follows that $\lim (c^\ell, N^{\ell+}) \in \Gamma^\star\left(\lim \alpha^\ell\right)$. If $u_i'(c)$ is continuously differentiable at $\lim c_i^\ell(\omega)$, then by continuity we have:

$$\lim \left(\alpha_i^\ell u_i' \left[c_i^\ell(\omega)\right] c_i^\ell(\omega)\right) = \left(\lim \alpha_i^\ell\right) \times u_i' \left[\lim c_i^\ell(\omega)\right] \times \left(\lim c_i^\ell(\omega)\right).$$

If $u_i(c)$ is not continuously differentiable at $\lim c_i^\ell(\omega)$ then given our maintained assumption that $u_i(c)$ is continuously differentiable over $(0, \infty)$, it must be that $\lim c_i^\ell(\omega) = 0$ and $u_i'(0) = -\infty$. Since $\lim c_i^\ell(\omega) = 0$ is part of a social optimum, it must be that $\lim \alpha_i^\ell = 0$. But we know in this case from Proposition C.1 that

$$\lim \alpha_i^\ell u_i' \left[c_i^\ell(\omega)\right] c_i^\ell(\omega) = 0 = \lim \alpha_i^\ell u_i' \left[\lim c_i^\ell(\omega)\right] \lim c_i^\ell(\omega).$$

Taken together, we obtain that $\lim \Delta^\ell = \lim \Delta^k \in \Delta^\star(\lim \alpha^\ell) = \Delta^\star(\lim \alpha^k)$.

Proof that $\Delta^\star(\alpha)$ is bounded. Otherwise, there would exists some sequence α^k and $\Delta^k \in \Delta^\star(\alpha^k)$ such that $\max |\Delta_i^k| \to \infty$. Since α^k belongs to a compact set we can extract a converging subsequence α^ℓ. Since $\Delta^\star(\alpha)$ has a closed graph $\lim \Delta^\ell \in \Gamma^\star(\lim \alpha^\ell)$ and so must be finite, which is a contradiction.

An auxiliary fixed-point problem. Let M be such that $\max |\Delta_i| \leq M$ for all $\Delta \in \Delta^\star(\alpha)$ and $\alpha \in \mathcal{A}$. Let $\mathcal{D}$ be the set of transfers $\Delta = (\Delta_1, \ldots, \Delta_I)$ such that $\sum_{i \in I} \Delta_i = 0$ and $\max |\Delta_i| \leq M$. Finally, let $K(\alpha, \Delta)$ be the function $\mathcal{A} \times \mathcal{D} \to \mathcal{A}$ such that

$$K_i(\alpha, \Delta) = \frac{(\alpha_i - \Delta_i)^+}{\sum_{k \in I} (\alpha_k - \Delta_k)^+},$$

where x^+ denotes the positive part of x. For each $(\alpha, \Delta) \in \mathcal{A} \times \mathcal{D}$, let the set $\Phi(\alpha, \Delta)$ be the product of the singleton $\{K(\alpha, \Delta)\}$ and the set $\Delta^\star(\alpha)$. By construction, $\Phi(\alpha, \Delta) \subseteq \mathcal{A} \times \mathcal{D}$. Since $\sum_{k \in I} (\alpha_k - \Delta_k)^+ \geq \sum_{k \in I} (\alpha_k - \Delta_k) = 1 > 0$ it follows that $K_i(\alpha, \Delta)$ is a continuous function over $\mathcal{A} \times \mathcal{D}$. Given our earlier result that $\Delta^\star(\alpha)$ has a closed graph, this implies that the correspondence $\Phi(\alpha, \Delta)$ has a closed graph as well. This allows to apply Kakutani's fixed point Theorem (see Corollary 17.55 in Aliprantis and Border (1999)) and assert that Φ has a fixed point, i.e., there exists some $(\alpha, \Delta) \in \mathcal{A} \times \mathcal{D}$ such that

$$\begin{aligned} \alpha_i &= \frac{(\alpha_i - \Delta_i)^+}{\sum_{k \in I} (\alpha_k - \Delta_k)^+} \text{ for all } i \in I \\ \Delta &\in \Delta^\star(\alpha). \end{aligned}$$

Proof that all fixed-points are such that $\Delta_i = 0$ for all $i \in I$. Next, we show that a fixed point of Φ has the property that $\Delta_i = 0$ for all $i \in I$. Indeed if $\alpha_i = 0$, then from the definition of $\Delta^\star(\alpha)$ we have that $\Delta_i \leq 0$, and from the fixed-point equation that $(-\Delta_i)^+ = 0 \Leftrightarrow \Delta_i \geq 0$. Hence, if $\alpha_i = 0$, then $\Delta_i = 0$. If $\alpha_i > 0$, then from the fixed point equation

$$\alpha_i \times \sum_{k \in I} (\alpha_k - \Delta_k)^+ = \alpha_i - \Delta_i \Rightarrow \Delta_i = \alpha_i \times \left[1 - \sum_{k \in I} (\alpha_k - \Delta_k)^+\right].$$

Hence, all Δ_i such that $\alpha_i > 0$ have the same sign. Since $\Delta_i = 0$ when $\alpha_i = 0$, it follows that all Δ_i have the same sign. But since $\sum_{i \in I} \Delta_i = 0$, this implies that $\Delta_i = 0$ for all $i \in I$.

E.7 Modified Security Market Line

Proposition E.1 *Suppose the distribution of tree supplies is strictly increasing. Let* $R_j(\omega) = \frac{d_j(\omega)}{p_j}$ *be the return of tree* j, $R_m(\omega) = \int_0^1 \frac{p_j}{\int_0^1 p_\ell d\bar{N}_\ell} R_j(\omega) d\bar{N}_j$ *the market return, and* $\beta_j = \frac{Cov(R_m, R_j)}{V(R_m)}$ *the market beta of tree* j. *Then,* β_j *is a continuous and strictly decreasing function of* j. *Moreover, the expected return of tree* j *is a piecewise linear function of* β_j:

$$\mathbb{E}[R_j - R_f] = \beta_j \Big(\mathbb{E}[R_m - R_f] - \theta_m\Big) + \theta_j, \tag{61}$$

where

$$\theta_j = \theta_k - \phi \max(\beta_j - \beta_k, 0\big) - \psi \max(\beta_k - \beta_j, 0\big), \tag{62}$$

and $R_f = \big(\sum_{\omega\in\Omega} q(\omega)\big)^{-1}$ *is the risk-free rate,* $\theta_j = \Delta_j/p_j$*, is the (per dollar invested) discount induced by incentive constraints for tree* j*,* k *is the marginal tree,* $\phi > 0$*,* $\psi > 0$*, and* $\theta_m = \int_0^1 \frac{p_j}{\int_0^1 p_\ell d\bar{N}_\ell}\theta_j d\bar{N}_j$ *is the average discount induced by incentive constraints. Equation (61) also holds for financial trees by setting* $\theta_j = 0$*.*

Proof that $j \mapsto \beta_j$ is strictly decreasing. Since there are only two states of nature, correlations are either equal to one, zero, or minus one. It follows from $R_m(\omega_1) < R_m(\omega_2)$ that $\beta_j = \frac{\sigma(R_j)}{\sigma(R_m)}\text{Sign}[d_j(\omega_2) - d_j(\omega_1)]$, where:

$$\Big(\sigma(R_j)\Big)^2 = \sum_{\omega\in\Omega}\pi(\omega)\left(\frac{d_j(\omega)-\bar{d}_j}{p_j}\right)^2 = \sum_{\omega\in\Omega}\pi(\omega)(1-\pi(\omega))^2\left(\frac{d_j(\omega_2)-d_j(\omega_1)}{p_j}\right)^2$$

Equation (11) implies that $p_j = a_i(\omega_1)d_j(\omega_1) + a_i(\omega_2)d_j(\omega_2)$, where i denotes the agent holding tree j and $a_i(\omega) > 0$. Thus:

$$\beta_j = \frac{1}{\sigma(R_m)}\left(\sum_{\omega\in\Omega}\pi(\omega)(1-\pi(\omega))^2\right)^{\frac{1}{2}} \frac{\frac{d_j(\omega_2)}{d_j(\omega_1)} - 1}{a_i(\omega_1) + a_i(\omega_2)\frac{d_j(\omega_2)}{d_j(\omega_1)}}. \tag{63}$$

$\frac{d_j(\omega_2)}{d_j(\omega_1)} \mapsto \beta_j$ is clearly continuous away from the marginal tree k. And it is also continuous at the marginal tree since p_j is continuous at $j = k$. For $j \neq k$, we can take the derivative:

$$\frac{d\beta_j}{d\frac{d_j(\omega_2)}{d_j(\omega_1)}} = \frac{1}{\sigma(R_m)}\left(\sum_{\omega\in\Omega}\pi(\omega)(1-\pi(\omega))^2\right)^{\frac{1}{2}} \frac{a_i(\omega_1)+a_i(\omega_2)}{\Big(a_i(\omega_1) + a_i(\omega_2)\frac{d_j(\omega_2)}{d_j(\omega_1)}\Big)^2} > 0.$$

Proof of equation (61). There is a different pricing kernel for each agent. For trees j held by agent i, the pricing kernel is:

$$1 = \mathbb{E}\left[\frac{q(\omega)}{\pi(\omega)}R_j(\omega)\right] - \delta\frac{\mu_i(\omega_i)}{\lambda_i}R_j(\omega_i).$$

Denoting the risk-free rate as $R_f = \big(E\big[\frac{q(\omega)}{\pi(\omega)}\big]\big)^{-1}$, the usual manipulations lead to:

$$\mathbb{E}[R_j(\omega) - R_f] = -R_f Cov\left(\frac{q(\omega)}{\pi(\omega)}, R_j(\omega)\right) + \theta_j,$$

where $\Delta_j = R_f\delta\frac{\mu_i(\omega_i)}{\lambda_i}R_j(\omega_i)$. Since there are two states of nature, $\frac{q(\omega)}{\pi(\omega)}$ can be written as an affine function of the market return with slope κ. Thus:

$$\mathbb{E}[R_j(\omega) - R_f] = -\kappa R_f Cov\big(R_m(\omega), R_j(\omega)\big) + \theta_j, \tag{64}$$

where $\theta_j = R_f\delta\frac{\mu_i(\omega_i)}{\lambda_i}R_j(\omega_i) = \frac{\Delta_j}{p_j}$. Multiplying by $\frac{p_j}{\int_0^1 p_\ell d\bar{N}_\ell}$ and integrating over j, we obtain the pricing kernel for the market portfolio:

$$\mathbb{E}[R_m(\omega) - R_f] = -\kappa R_f Var\big(R_m(\omega)\big) + \theta_m, \tag{65}$$

where $\Delta_m = \int_0^1 \frac{p_j}{\int_0^1 p_\ell d\bar{N}_\ell}\theta_j d\bar{N}_j$. Combining (64) and (64) yields the modified CAPM formula (61).

Next, we show that θ_j can be written as a piecewise linear function of β_j with a kink at the marginal tree β_k. $R_j(\omega_1) = \frac{d_j(\omega_1)}{p_j} = \frac{1}{a_i(\omega_1)+a_i(\omega_2)b_j}$, where i denotes the agent holding tree j and $b_j \equiv \frac{d_j(\omega_2)}{d_j(\omega_1)}$. Equation (63) implies that β_j can be written as a function of b_j: $\beta_j = \rho_0\frac{b_j - 1}{a_i(\omega_1)+a_i(\omega_2)b_j}$, where $\rho_0 = \frac{1}{\sigma(R_m)}\big(\sum_{\omega\in\Omega}\pi(\omega)(1-\pi(\omega))^2\big)^{\frac{1}{2}}$. Inverting this function, we can write b_j as a function of β_j: $b_j = \frac{\rho_0+\beta_j a_i(\omega_1)}{\rho_0-\beta_j a_i(\omega_2)}$. Thus: $R_j(\omega_1) = \frac{\rho_0-\beta_j a_i(\omega_2)}{(a_i(\omega_1)+a_i(\omega_2))\rho_0}$. Similarly: $R_j(\omega_2) = \frac{\rho_0+\beta_j a_i(\omega_1)}{(a_i(\omega_1)+a_i(\omega_2))\rho_0}$. It implies that Δ_j is linear and decreasing in β_j for trees j held by agent 1 and linear and increasing for trees held by agent 2. It follows from the continuity of θ_j at the marginal tree k that

θ_j can be written as (62).